EXACT VIBRATION CONTROL AND BOUNDARY STABILIZATION OF A HYBRID INTERNALLY DAMPED ELASTIC STRUCTURE

THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY (SCIENCE)
OF
JADAVPUR UNIVERSITY

GANESH CHANDRA GORAIN

SATYENDRA NATH BOSE NATIONAL CENTRE
FOR BASIC SCIENCES, CALCUTTA, INDIA

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सत्येन्द्र नाथ बसु राष्ट्रीय मौलिक विज्ञान केन्द्र
SATYENDRA NATH BOSE NATIONAL CENTRE FOR BASIC SCIENCES

CERTIFICATE

This is to certify that the thesis entitled **EXACT VIBRATION CONTROL AND BOUNDARY STABILZATION OF A HYBRID INTERNALLY DAMPED ELASTIC STRUCTURE** submitted by Sri **GANESH CHANDRA GORAIN** who got his name registered on **23. 02. 1998** for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own work under the Supervision of **Prof. Sujit K. Bose** and that neither this thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before.

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LIST OF PUBLICATIONS AND PREPRINTS

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2. "Exact Controllability and Boundary Stabilization of Torsional Vibrations of an Internally Damped Flexible Space Structure", G. C. Gorain and S. K. Bose, *Journal of Optimization Theory and Applications,* Vol. **99**, pp. 423-442, (1998).

3. "Exact Controllability of a Linear Euler-Bernoulli Panel", G. C. Gorain and S. K. Bose, *Journal of Sound and Vibration,* Vol. **217**, pp. 637-652, (1998).

4. "Stability of the Boundary Stabilised Internally Damped Wave Equation $y'' + \lambda y''' = c^2(\Delta y + \mu \Delta y')$ in a Bounded Domain in $\mathbf{R}^n$ ", S. K. Bose and G. C. Gorain, *Indian Journal of Mathematics,* Vol. **40**, pp. 1-15, (1998).

5. "Exact Controllability and Boundary Stabilization of Flexural Vibrations of an Internally Damped Flexible Space Structure", G. C. Gorain and S. K. Bose, Communicated to *Applied Mathematics and Computation.*

6. "Boundary Stabilization of a Hybrid Euler-Bernoulli Beam", G. C. Gorain and S. K. Bose, Communicated to *Proceedings Indian Academy of Sciences (Mathematical Sciences).*

CONTENTS

INTRODUCTION

Motivated by ambitious space programs, vibrations of large scale flexible space structures have been an active area of research among others. Such structures are usually comparatively light weight and as a consequence are likely to be flexible. Due to building of larger and more flexible structures, the analysis and design of flexible mechanical systems has steadily gained in importance. The approach has been to study a prototype model, such as, a long flexible rectangular panel attached to a movable rigid body (hub) at one end to represent a Solar Cell Array. In general moderate disturbance may result in moderate frequency vibration in the system due to its flexibility and the most important problem for these flexible structures is to suppress the vibrations to assure a good performance.

Mechanical systems of the above prototype consisting of coupled elastic part and rigid part, constitute the class of hybrid systems. Such systems occur elsewhere also, such as in robots with flexible arms, spacecraft with flexible appendages or a flexible beam (mast) joining two rigid bodies—one representing Space Shuttle Orbiter considered fixed and the other representing the Antena Reflector. In fact, the hybrid systems encompass a rather extensive class and the investigation to control and stabilize their vibrations—are frontier area of research for theoretical as well as experimental understanging, forming the substance of this Thesis.

The discrete coordinate system can be a convenient practical approach to describe the dynamical-response analysis of a space structure. Due to representation of motion by a limited number of displacement coordinates, the formal action can only approximate the actual dynamical behavior. The precision of the result towards the actual behavior can of course be improved by increasing the number of degrees of freedom infinitely. In mathematical formalism, the dynamical consideration of a system of infinitely many connected points leads to differential equations in which the position coordinate acquires the role of an independent variable. In as much as the time is also an independent variable like the position coordinate, the formulation of the equations of motion in this way leads to partial differential equations with two independent variables (time t and space coordinate x). The system is thus commonly known as distributed parameter system.

The vibrations of flexible space structures are thus problems of dynamcal system theory governed by partial differential equations. These types of problems are very significant from mathematical point of view also. Mechanical engineers are primarily interested in structural analysis that means, determination of stress and displacement distributions under prescribed loads and constraints; free and forced vibrations that means, computation of nodes and natural frequencies; problem of parametric excitation and elastic stability (buckling of columns and plates) and optimal structural design of elastic system (cf. Clough and Penzien [17], Meirovitch [60], Timoshenko and Gere [84]). On the other hand, some possible objectives of control engineers are the determination of distributed or boundary value control variable (loads, frequency, bending moments etc.) to obtain a desired dynamic behavior of the elastic system under consideration and application of Lyapunov's stability theory (cf. LaSalle and Lefschetz [45]) for solving kinematic stability problem (Timoshenko *et al.* [85], Ray and Lainiotis [75]).

A very common approach to treat the vibration problem in engineering literature, is to decompose the vibration into normal modes and retain the first few modes to reduce the problem to finite dimensional state-space representation governed by ordinary differential equations (cf. Fukuda *et al.* [22,23,24], Bontsema *et al.* [3], Matsuno *et al.* [59], Sakawa and Luo [79]). This approach is termed 'modal' analysis. However, considering the dynamic response for this reduced finite set of equations does not always guarantee that the same response will work on the original set of equations. In fact, since the actual number of modes of an elastic system is infinite, the number of modes that should be retained is not known *a priori*.

Research in the area of exact controllability and boundary stabilization problems for distributed parameter systems have been developing in a significant manner. The most common classes of vibration control mechanisms are of passive, active and of hybrid type. Passive vibration control uses resistance devices that absorb vibration energy. Active vibration control is also like that, but involves the use of force actuators linked with external energy. Hybrid vibration control is a combination of passive approach with active control. The objective is to bring the vibrations to null in some finite time T. From mathematical point of view, exact controllability at time $T > 0$ of a system states the capability, using a suitable control function defined in $[0, T]$, to steer the system from an arbitrary initial state say, (u_0, v_0) in some Hilbert space H to an equally arbitrary desired final state say, (u_T, v_T) in H at the time T. In particular, when the final state $(u_T, v_T) = (0, 0)$, the corresponding property is termed as null controllability and when the initial state $(u_0, v_0) = (0, 0)$, the corresponding problem is called reachability. Approximate controllability is usually stated in the context of approximate reachability, the ability to drive the system from some initial state (u_0, v_0)

to the set of final state (u_T, v_T) dense in the space H. The associated type of stability most commonly studied in the mathematical literature are strong stability and uniform stability. A system is called strongly stable, if the energy $E(t)$ of each solution of the system converges to zero as time $t \to +\infty$. If the convergence is uniform for $t > 0$ with respect to all initial data in the energy space for which $E(0) < \infty$, the system is called uniformly stable. If the stability property can be achieved due to incorporation of a stabilizer or a damping device applied on the boundary, the system is then called boundary stabilizable.

Research on problems of controllability and stability for distributed parameter systems has started in an extensive way dating back to the seventies (cf. Russell [77,78], Graham and Russell [31], Lagnese [40,41], Chen [6,7,8]) and the idea was first applied to the flexible system governed by wave equation. The theory of exact controllability for second order hyperbolic equation with standard boundary conditions has been studied by several authors (cf. Chen [6], Lions [52], Ho [32], Zuazua [89]), and they have commented on performance limitations. To study exact controllability, a systematic method named, HUM for 'Hilbert Uniqueness Method', based on uniquess results and on Hilbert spaces constructed by using uniqueness, avoiding normal modes altogether has been introduced by Lions [52], for distributed systems governed by second order wave equation and fourth order Petrowski equation with Dirichlet and Neumann boundary conditions. In contrast, Chen [6] has achieved the exact controllability result for wave equation in a bounded domain by stability method. Ho [32] established it by means of a minimization problem corresponding with its adjoint system and has shown the observability of the adjoint system by multiplier technique. He considered one dimensional wave equation with variable wave speed and locally distributed control as a model and later in the paper [33], he obtained a case of approximate controllability of the system by treating a problem of coupled strings with control applied at the coupled points. Zuazua [89] studied the semilinear wave equation in the bounded domain with both Dirichlet and Neumann boundary conditions to obtain the exact controllability result by HUM while, by dynamic programming arguments, a strongly damped wave equation with Dirichlet boundary condition was treated by Bucci [5].

There has been extensive work in the last decade on boundary stabilization of wave equation (cf. Chen [8,9], Lagnese [42,43], Lasiecka and Triggiani [46], Komornik [37], Komornik and Zuazua [38]). In summary, the pioneering work was first started since early sixties in a study aimed at achieving energy decay rates for the wave equation exterior to a bounded obstacle (the so called exterior problems) (cf. Morawetz [63], Lax *et al.* [50]), and investigations had been continuing till the mid-seventies (cf. Morawetz *et al.* [64], Quinn and Russell [70], Strass [83]). These efforts brought forward several energy identities, which were then used to obtain energy decay rates under suitable geometrical conditions on the boundary of the obstacle. On the other hand, the study of the analogous problem in boundary domains with an 'energy absorbing' boundary began in the seventies (cf. Russell [77], Rauch and Taylor [74], Slemord [82]). The latter kind of problem is more difficult than the exterior problem, since the latter enjoys the advantage that the energy distributes itself over an infinite region as time $t \to \infty$. To investigate the latter kind of problem, Chen [6] first obtained the energy decay rates (uniform stabilization) for the interior problem, under some natural geometrical conditions on the domain, by adapting the multiplier technique developed earlier for the exterior problem. His later paper [7] relaxed the geometrical conditions on the domain by employing this time a new energy functional discovered by Strass [83] in the study of exterior problem. Later considering some energy functional, Lagnese [42] managed to relax even further the geometrical conditions on the domain under which an energy decay rate is obtained. Using one dimensional wave equation with distributed viscous damping as a model, Chen *et al.* [10] has shown an asymptotic average decay rate of eigen modes equal to the damping rate of the high frequencies of the wave. The result of uniform exponential energy decay estimates for the solution of wave equation in a bounded domain has been obtained directly by Lagnese [43], Komornik and Zuazua [38] by suitable viscous boundary feedback and subsequently using a special kind of feedback, Komornik [37] obtained faster energy decay rates.

Chen *et al.* [12] treated the problems of two coupled vibrating strings with a stabilizer or damping device installed at the coupling point to achieve uniform as well as non-uniform exponential decay property of vibration energy, subject to some restrictions on the arrangement of the stabilizer and the wave velocities of the strings. For coupled vibrating strings, the difficulty stemmed from different wave speeds in each of them. With a stabilizer installed at the coupling point, this was successfully tackled by Liu in [56], using frequency domain methods and in his later paper [57], the idea has been extended to a long chain of coupled vibrating strings, where a stabilizer is installed at each internal node and also at the boundary. It is proven that the energy of the system decays uniformly exponentially if there is a stabilizer installed at the boundary point. If the stabilizers are installed only at internal nodes, the energy may decay either uniform exponentially or non-uniformly, or may not be decay at all, depending on the different wave speeds and the stabilizer arrangement. The idea was then applied by Kim [35] to a composite bar consisting of two different segments, only one of which is damped. The case of serially connected strings has been treated in Lee and You [51]. Chen *et al.*

[11] have presented a general theorem on an abstract evolution equation with partially distributed damping in which the conditions for exponential decay of energy involve eigen values and eigen functions of the associated stationary system.

Subsequently, the idea of controllability and stability for the flexible elastic structures was extended to systems governed by Euler-Bernoulli beam equation (cf. Chen *et al.* [13], Lasiecka and Triggiani [47,48], Krall [39], Morgül [66,67]). In particular, in Chen *et al.* [13], it has been proven that, in a cantilever beam, a single non-dynamic actuator applied at the free end of the beam is sufficient to uniformly stabilize the beam vibrations. Two boundary controls are taken into account to study exact controllability for Euler-Bernoulli beam problem in a bounded subspace in the work of Lasiecka and Triggiani [47], one for Dirichlet and other for Neumann boundary control, and in their [48], one for displacement and other for moment boundary control. On the other hand, using the basic principal of HUM, Kim [36] has established the exact semi-internal controllability of an Euler-Bernoulli beam with variable coefficients. The uniform stability of the vibrations of a flexible structure, described by one dimensional Euler-Bernoulli beam, clamped at one end has been treated by Krall [39], with boundary controls at the other end and later the same has been treated by Morgül [66], with dynamic boundary control at that end. The result is then extended for rotating flexible structure in Morgül [67], using energy functional technique. A good source of reference for boundary stabilization of various plate models is the work of Lagnese [44].

Controllability and Stability of the vibrations of elastic systems, particularly flexible structures have been studied in the past. The appearence of hybrid vibrating systems is rather common. A rigorous dynamic model of flexible space craft, in the form of a hybrid system was first introduced by Meirovitch [61,62]. To obtain stabilization of the system Lyapunov's approach was used. Similar problems were later treated by Biswas and Ahmed [2] and some simple feedbacks were suggested for stabilization. The exact controllability and boundary stabilization of a hybrid system of elastic vibrations of an elastic beam linked at its boundary to a rigid mass was demonstrated in Littman and Markus [54,55]. These papers are devoted to the control design and stabilization of SCOLE (Spacecraft Control Laboratory Experiment) model—one specific elastic structure that forms a basic type of component in many more complicated and extensive space-environment constructions. In one specific case, Littman and Markus [55], prove the strong stabilization together with the lack of uniform stabilization for the hybrid model of vibrations. This type of hybrid control of an elastic structure was generlized to two dimensional rectangular elastic plate by You [87,88], with inertial properties along the control edge, which is rimmed with a flange of lip. The idea is extended by Markus and You [58] to the problem of elastic plates with suitable boundary conditions at the three clamped sides and dynamical boundary control from the free side to obtain an approximate control system. Recently, using a method of compact perturbation, Rao [73] has generalized the result of Littman and Markus [55], in the case the hybrid clamped beam with an end mass, and with usual boundary feedback applied to the end with the mass. All their investigations have shown the controllability or stability of Euler-Bernoulli beam or wave equation, clamped at one end and free at the other, except for feedback damping or control force.

For hybrid system, which is our main concern in this Thesis, it is important from practical point of design as well as theoretical challenge, that most interest lie in the boundary control, pointwise control or even pointwise boundary control. Most of the work in this area has dealt with the problems of boundary control applied at free end and clamped on the other end. Actually for this class of system, the most common practical problem generally occurs when it is very difficult or undesirable to apply the boundary control at the free end of the elastic part where as to apply it on the rigid part is easier to obtain a good performance of the overall system. This type of hybrid problems are very significant from mathematical point of view.

In this Thesis, the exact controllability and boundary stabilization of hybrid elastic structure composed of a rectangular flexible panel and with a rigid hub at one end, totally free at the other end will be analyzed and investigated systematically by a control force or torque applicable on the rigid hub only. Among the various types of vibrations of elastic systems, the main types of vibration viz., the rotational or torsional vibrations and transverse or flexural vibrations of beams or plates are considered separately to study the exact controllability and boundary stabilization of the system. Basically, these two types of vibrations under consideration are mathematically represented by the linearized equations

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} \tag{0.1}$$

$$m\frac{\partial^2 y}{\partial t^2} + EI\frac{\partial^4 y}{\partial x^4} = 0 \tag{0.2}$$

respectively. In the torsional vibration equation (0.1), known as the wave equation, ϕ denotes the rotational deflection at any point (x, t) and the parameter c the torsional wave velocity, depending on the torsional rigidity of the material and the radius of gyration about the the neutral axis of the structure. In the flexural vibration equation (0.2), known as the Euler-Bernoulli beam equation, y denotes the transverse deflection at any point (x, t) and the parameters m, EI respectively the mass per unit length and flexural rigidity of the structure.

We know that all structural vibration are attenuated, more or less quickly by internal dissipative mechanism, unless some external forces provide continued excitation. In endeavour we intend to incorporate certain simple models incorporating internal dissipation or material damping. Our criteria for admitting internal damping term includes correspondence with physical reality as evident by experiment and well-posedness from mathematical point of view. In part to account this progressively, we consider the damping mechanism commonly referred to as the Kelvin-Voigt damping (dashpot in series with a spring), after prominant English and German physicists of last century. For this type of damping it is supposed that the dissipative forms are obtained from the velocity in essentially the same way as restoring forces are obtained from displacement. In the case of the torsional and flexural vibrations the approximate equations (0.1) and (0.2) respectively become

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} + \mu \frac{\partial^3 \phi}{\partial x^2 \partial t} \tag{0.3}$$

and

$$m \frac{\partial^2 y}{\partial t^2} + EI\Big(\frac{\partial^4 y}{\partial x^4} + \mu \frac{\partial^5 y}{\partial x^4 \partial t}\Big) = 0 \tag{0.4}$$

where $\mu > 0$ is the internal damping parameter. In the literature (cf. Christensen [16]) more complicated models of internal damping have been proposed, but the Kelvin-Voigt model has been chosen to make the treatment as simple as possible.

For the sake of simplicity and easy readability, we devide the whole work of the Thesis into two parts. In Part I, we have demonstrated the exact controllability of torsional vibration as well as flexural vibration of a hybrid system consisting of an elastic rectangular panel with a rigid movable hub at one end and totally free at the other (such as solar cell array). An active control force or torque is applied on the rigid hub of the panel to suppress the vibrations exactly, when motion is set from given initial displacement and velocity along the length of the panel. We have studied the exact controllability after making the problems more realistic by incorporating small internal material damping of the structure as alluded above. In Part II, we have presented uniform boundary stability results of these problems by means of exponential energy decay estimates of the form

$$E(t) \leq M e^{-\beta t} E(0), \qquad t \geq 0 \tag{0.5}$$

for some reals $M \geq 1$ and $\beta > 0$, by employing a viscous damping at the hub end only. In this part, we have also worked on the mathematical problem of uniform stability for the solution of internally damped wave equation

$$y'' = \Delta y + \mu \Delta y' \tag{0.6}$$

and with a more complicated ***standard linear model*** of viscoelasticity for internal damping,

$$y'' + \lambda y''' = c^2(\Delta y + \mu \Delta y') \tag{0.7}$$

in a bounded domain Ω in $\mathbf{R}^n$ with piecewise smooth boundary Γ $(0 < \lambda < \mu)$ to obtain the result (0.5) in explicit form, where prime denotes the differentiation with respect to time and Δ, the Laplacian in $\mathbf{R}^n$ taken in the space variables.

The outline of this Thesis, divided into two parts, is as follows:

Part I, comprises the exact controllability problems and consists of five Chapters, namely, Chapters 1 to Chapter 5; while Part II comprises uniform stability problems and consists of six Chapters namely, Chapter 6 to Chapter 11. In fact, Chapter 6 to Chapter 9 involve the boundary stability problems relating to Chapters 1 to Chapter 4 and Chapter 10 to Chapter 11 involve two mathematical uniform stability problems of generalized viscoelastic damping.

In Chapter 1, we have made a detailed discussion of the exact controllability of torsional vibrations of a rectangular panel hoisted by a rigid hub at one end. We first formulate the physical problem into mathematical analogy. Due to installation of the hub at one end of the panel, its dynamics leads to a nonstandard boundary condition and as a whole the system becomes a hybrid system of dynamics. It is shown that the whole system is exactly controllable for a time $T > 2l/c$, where l is the length of the panel and c the torsional wave velocity, by means of an active control torque applied on the rigid hub only. The technique used to establish the exact controllability result is the Hilbert Uniqueness Method. The mathematical development is given in detail, so that the techniques are easily comprehensible in the subsequent chapters.

In Chapter 2, we have incorporated small internal damping of the material to the governing equation of the problem discussed in Chapter 1, to achieve a more realistic model from practical point of view. The internal damping is modeled according to Kelvin-Voigt viscoelasticity. In this case also we have established the exact controllability result using HUM, by means of a suitable boundary control torque applied on the rigid hub only. An estimated minimum time for exact controllability for the corresponding system has also been obtained theoretically.

In Chapter 3, we have made a detailed discussion of the exact controllability of flexural vibrations of the hybrid structure consisting of a rectangular flexible panel with a rigid hub at one end. After presenting the mathematical formulation of the problem, using HUM the exact controllability result as well as the minimum time of exact controllability, is achieved by means of an active boundary control force applied on the rigid hub.

In Chapter 4, we have included small internal damping of the material based on Kelvin-Voigt model, to the problem considerd in Chapter 3, so that from physical point of view the problem becomes more realistic. It is shown that vibration of the overall system can be driven to rest by means of a suitable boundary control force applied on the rigid hub only. Also an estimate of the minimum time of control is obtained theoretically by HUM.

In Chapter 5, we have constructed a closed form appproximate numerical scheme for the problem discussed in Chapter 3, by Galerkin's residual technique to support and implement the method from practical point of view. An approximate closed form solution together with approximate boundary control are elicitted from the scheme.

In Chapter 6, we have made a detailed discussion on boundary stabilization for the torsional vibrations of the hybrid system presented in Chapter 1. The uniform exponential decay of energy of the form (0.5) is obtained directly for the solution of such formulation for torsional vibrations of the panel with viscous boundary damping at the hub end only.

In Chapter 7, we have presented the boundary stability result for internally damped torsional vibration problem described in Chapter 2. We have achieved explicitly the uniform exponential energy decay estimate for the corresponding solution of the problem, by considering only a viscous boundary damping at the hub end.

In Chapter 8, we have discussed the uniform boundary stability for the flexural vibrations of the hybrid system described in Chapter 3. An explicit form of uniform exponential energy decay rate is established for the solution of such problem for small vibrations of the panel with a viscous boundary damping at the hub end.

In Chapter 9, we have made an illustration on boundary stabilization for the internally damped flexural vibrations problem described in Chapter 4. The uniform decay of solution by means of exponential energy decay estimate is substantiated directly for such formulation by employing a viscous boundary at the hub end only.

In Chapter 10, we have made a detailed demonstration on internally damped wave equation (0.6) in a bounded domain Ω in $\mathbf{R}^n$ with piecewise smooth boundary Γ under mixed undamped boundary conditions. We have briefly expressed the surrounding literatures and objective of presentation. An uniform exponential energy decay rate of the form (0.5) is explicitly established for the solution of this type of boundary value problem.

In Chapter 11, we have manifested the boundary stability of internally damped wave equation of the form (0.7) in a bounded domain Ω in $\mathbf{R}^n$ as a more realistic form by the treatment of the model equation as 'standard linear model' of viscoelasticity. We have briefly discussed the physical motivation behind it. The uniform exponential energy decay rate of the form (0.5) is explicitly obtained subject to mixed boundary conditions.

Finally, in Chapter 12, we have presented the concluding discussion where we have summarized the objectives and achievements of this work. We have also deliberated upon the strength of the work, and briefly focussed the scope of further work.

Part I – EXACT CONTROLLABILITY

CHAPTER 1

EXACT CONTROLLABILITY OF TORSIONAL VIBRATIONS OF A FLEXIBLE PANEL

1.1 Introduction

Mathematical control theory of distributed parameter systems is currently under extensive study, in view of application of vibration control of various structural elements. The torsional vibrations of elastic space structures are the problems of dynamical system governed by partial differential equations. The problems of controllability and stability of second order hyperbolic equation with standard boundary conditions has been studied theoretically by a number of authors (cf. Chen [6,7,8], Lions [52], Morgül [65]) and have been commented on the performances. To study exact controllability, a systematic method named HUM (Hilbert Uniqueness Method), has been introduced by Lions [52] for distributed parameter systems, governed by the second order wave equation and the fourth order Petrowsky equation with Dirichlet and Neumann boundary conditions. In contrast, taking into account only the first few modes of vibrations, after decomposition into normal modes, to reduce the vibration problem into finite dimensional state-space form, is a very common approach for treatment in engineering literature (cf. Fukuda *et al.* [22,23,24]).

In this Chapter, we formulate a distributed hybrid dynamical model of torsional vibrations of a large flexible space structure such as, solar cell array and examine the exact controllability of the dynamics of the system. This type of model usually consist of coupled elastic and rigid parts, constituting a hybrid system. Due to complexity of the dynamics and need to control with high-quality performance of the whole system, it is necessary to apply suitable control force or torque on the system. Earlier, Littman and Markus [54,55] and later Rao [72,73] treated this type of problem and have shown the ability to control or stabilize vibrations modeled by wave equation or Euler-Bernoulli beam equation, clamped at one end and control force or feedback damping applied on the other end. But application of control force or feedback damping on that end of the elastic part is very difficult or undesirable in practice.

As a simple model, here we consider the problem of torsional vibrations of an elastic rectangular panel hoisted by a movable rigid hub at one end and totally free at the other end. The dynamics of vibrations is governed by one dimensional wave equation. Installation of the hub at one end of the panel leads to a nonstandard boundary condition and as a whole the system becomes a hybrid model. An active control force is applied only on the rigid hub to suppress the vibrations of the system exactly following prescribed initial motion, without applying any constraint at the free end. The investigation has also put forward an estimated least time of exact controllability in the framework of HUM.

1.2 Mathematical Formulation

The physical model to be considered here is a simple type of structure consisting of a uniform rectangular flexible panel of unit width and length l held at one end by a rigid hub of mass m_h and totally free at the other end. The hub end considered as lumped mass is capable of motion under the action of control. Our objective is to control the vibrations of this system exactly (to the case of null controllability), by application of a suitable active control torque $Q(t)$ to the rigid hub, in some finite time interval [0,T] when it is initially set in motion.

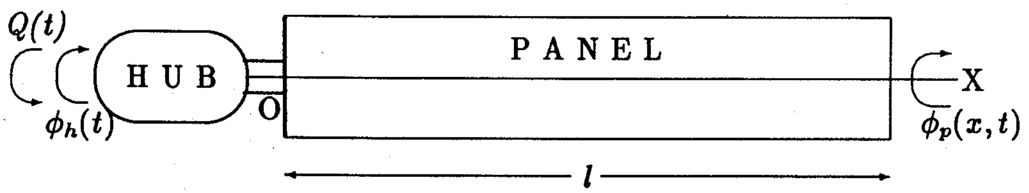

Figure 1.1. *Schematic of the rigid hub and the panel for torsional vibrations.*

Referring to the schematic Figure 1.1, if $\phi_h(t)$ is the rotation of the rigid hub and $\phi_p(x,t)$ that of the panel at the position x along the span of panel relative to the hub at time t, then the total rotational angle

$$\phi(x,t) = \phi_h(t) + \phi_p(x,t), \qquad 0 \le x \le l, \; 0 \le t \le T, \tag{1.1}$$

of the panel satisfies torsional vibrations equation

$$\ddot{\phi}(x,t) = c^2\phi''(x,t), \qquad 0 \le x \le l, \; 0 \le t \le T, \tag{1.2}$$

under the assumption that rate of change of ϕ along the length of the panel i.e., $|\phi'(x,t)|$ is very small, where dots and primes denote differentiation with respect to time coordinate t and space coordinate x respectively, and $c^2 = D_p/\rho_p J_p$. The constants D_p, ρ_p, J_p are respectively the torsional rigidity, the density and the radius of gyration about the central axis of the panel.

Initially at time $t = 0$, the panel is set to vibrations with initial values

$$\phi(x,0) = \phi_0(x) \quad \text{and} \quad \dot{\phi}(x,0) = \phi_1(x), \qquad 0 \le x \le l. \tag{1.3}$$

At the hub end $x = 0$ where the control torque $Q(t)$ is applied, the equation of motion is

$$I_h\ddot{\phi}_h(t) = D_p\phi_p'(0,t) - Q(t) \tag{1.4}$$

where I_h is the total moment of inertia of the hub about its axis of rotation. Since at $x = 0$ we have $\phi_p(0,t) = 0$, yielding $\phi_h(t) = \phi(0,t)$. Also from equation (1.1), we have $\phi'(x,t) = \phi_p'(x,t)$. Equation (1.4) then becomes

$$\phi'(0,t) = \alpha\,\ddot{\phi}\,(0,t) + \lambda Q(t), \qquad 0 \le t \le T, \tag{1.5}$$

where $\alpha = I_h/D_p$ and $\lambda = 1/D_p$. The free end $x = l$ of the panel yields the condition

$$\phi'(l,t) = 0, \qquad 0 \le t \le T. \tag{1.6}$$

Therefore, the mathematical problem to be concerned for controllability of the torsional vibrations of the uniform rectangular panel as described above, is governed by the hybrid system:

$$\begin{aligned}
&\ddot{\phi}\,(x,t) = c^2\phi''(x,t) && && 0 \le x \le l, \; 0 \le t \le T, \\
&\phi(x,0) = \phi_0(x), && \dot{\phi}(x,0) = \phi_1(x), && 0 \le x \le l, \\
&\phi'(0,t) = \alpha\,\ddot{\phi}\,(0,t) + \lambda Q(t), && \phi'(l,t) = 0, && 0 \le t \le T.
\end{aligned} \tag{1.7}$$

To study the exact controllability of the mathematical problem (1.7) at some finite time $T > 0$, our aim is to find a control torque $Q(t)$ appropriately, such that $Q(t)$

drives the system (1.7) to rest (the desired final state) at time $t = T$. Then the solution of (1.7) must satisfy

$$\phi(x,T) = \dot{\phi}(x,T) = 0. \tag{1.8}$$

To establish (1.8) for the system (1.7) by HUM, some auxiliary results are needed which are in the following.

1.3 Adjoint System.

Associated with each solution of the system (1.7), we start with $\theta(x,t)$, the solution of its adjoint system:

$$\begin{aligned}
&\ddot{\theta}\,(x,t) = c^2\theta''(x,t), && && 0 \le x \le l,\ 0 \le t \le T,\\
&\theta(x,0) = \theta_0(x), && \dot{\theta}(x,0) = \theta_1(x), && 0 \le x \le l,\\
&\theta'(0,t) = \alpha\,\ddot{\theta}\,(0,t), && \theta'(l,t) = 0, && 0 \le t \le T,
\end{aligned} \tag{1.9}$$

under the assumption $\theta_0(0) = 0$, $\theta_1(0) = 0$. Now, for every $\{\theta_0, \theta_1\} \in F = L^2[0,l] \times H^{-1}[0,l]$, where $H^{-1}[0,l]$ is the dual space of the Sobolev space $H^1[0,l]$ of order one given by

$$H^1[0,l] = \left\{ f \mid f \in L^2[0,l],\ \frac{\partial f}{\partial x} \in L^2[0,l] \right\},$$

the system (1.9) has a unique solution $\theta(x,t)$ (cf. Lions and Magenes [53]) for $0 \le x \le l$, $0 \le t \le T$.

To each solution of (1.9), the total energy at time t is defined by

$$E(t) = \frac{1}{2}\int_0^l (\dot{\theta}^2 + c^2\theta'^2)dx + \frac{c^2\alpha}{2}\dot{\theta}^2(0,t). \tag{1.10}$$

Differentiating (1.10) with respect to t and using the first equation of (1.9), we have

$$\dot{E}(t) = \int_0^l c^2(\dot{\theta}\theta'' + \theta'\dot{\theta}')dx + c^2\alpha\dot{\theta}(0,t)\ddot{\theta}(0,t).$$

Integrating by parts we have,

$$\dot{E}(t) = c^2\left[\dot{\theta}\theta'\right]_0^l + c^2\alpha\dot{\theta}(0,t)\ddot{\theta}(0,t).$$

Applying the boundary conditions of (1.9), the above finally yields

$$\dot{E}(t) = 0, \tag{1.11}$$

which implies that

$$E(t) = \text{constant} = E(0), \qquad \text{for } 0 \le t \le T, \tag{1.12}$$

where,

$$E(0) = \frac{1}{2}\int_0^l \left(\dot{\theta}^2(x,0) + c^2\theta'^2(x,0)\right)dx \tag{1.13}$$

since $\theta_1(0) = 0$, and thus the adjoint system is energy conserving system.

1.4 Backward System and Operator Λ

We next consider $\psi(x,t)$, the solution of a time backward system:

$$\begin{aligned}
&\ddot{\psi}(x,t) = c^2\psi''(x,t) && 0 \le x \le l,\ 0 \le t \le T,\\
&\psi(x,T) = 0, \qquad \dot{\psi}(x,T) = 0, && 0 \le x \le l,\\
&\psi'(0,t) = \alpha\ddot{\psi}(0,t) + \beta_0\lambda\theta(0,t), \qquad \psi'(l,t) = 0, && 0 \le t \le T
\end{aligned} \tag{1.14}$$

where β_0 is a constant independent of t. It is clear that the solution of the non-homogeneous boundary value problem (1.14) depends on $\theta(0,t)$, i.e., depends on the solution of the system (1.9), and hence on the initial values θ_0 and θ_1 of (1.9). Now for given $\{\theta_0,\ \theta_1\}$ in the Hilbert space F, the system (1.14) has a solution $\psi(x,t)$ (cf. Lions and Magenes [53]). After knowing $\psi(x,t)$, the functions $\psi(x,0)$ and $\dot{\psi}(x,0)$ can be obtained easily. We can therefore uniquely define an opertor Λ as

$$\Lambda\{\theta_0,\theta_1\} = \{\psi_1, -\psi_0\}, \tag{1.15}$$

where

$$\psi_0 = \psi(x,0) \quad \text{and} \quad \psi_1 = \dot{\psi}(x,0). \tag{1.16}$$

According to Lions [52], we shall now estimate the functional

$$\{\theta_0,\theta_1\} \to \left\langle \Lambda\{\theta_0,\theta_1\}, \{\theta_0,\theta_1\}\right\rangle$$

given by

$$\left\langle \Lambda\{\theta_0,\theta_1\}, \{\theta_0,\theta_1\}\right\rangle = \int_0^l (\theta_0\psi_1 - \theta_1\psi_0)dx. \tag{1.17}$$

For this, multiplying the first equation of (1.9) by ψ and that of (1.14) by θ, and integrating over $[0,l]\times[0,T]$ after subtraction, we obtain

$$\int_0^l\int_0^T \frac{\partial}{\partial t}(\dot{\theta}\psi - \theta\dot{\psi})dxdt = c^2\int_0^l\int_0^T \frac{\partial}{\partial x}(\theta'\psi - \theta\psi')dxdt.$$

Integrating by parts and applying the boundary and initial conditions of the systems (1.9) and (1.14), the above yields

$$\int_0^l (\theta_0\psi_1 - \theta_1\psi_0)dx = c^2\lambda\beta_0\int_0^T \theta^2(0,t)dt - c^2\alpha\int_0^T \left[\ddot{\theta}(0,t)\psi(0,t) - \ddot{\psi}(0,t)\theta(0,t)\right]dt.$$

Again integrating by parts, the second integral after simple calculations eventually reduces to zero, as $\theta_0(0) = 0$, $\theta_1(0) = 0$. Thus from above, we finally have

$$\int_0^l (\theta_0\psi_1 - \theta_1\psi_0)dx = c^2\lambda\beta_0 \int_0^T \theta^2(0,t)dt. \tag{1.18}$$

By (1.17) and (1.18), the functional is thus estimated as

$$\left\langle \Lambda\{\theta_0,\theta_1\},\{\theta_0,\theta_1\}\right\rangle = \int_0^l (\theta_0\psi_1 - \theta_1\psi_0)dx = C\int_0^T \theta^2(0,t)dt, \tag{1.19}$$

where, $C = c^2\lambda\beta_0$. For T large enough say, $T > T_0$, (T_0 is estimated in the next section), we shall show that the functional defined by (1.19) defines a norm on the initial values $\{\theta_0,\theta_1\}$, equivalent to the norm on the space F, i.e.,

$$\left|\left|\{\theta_0,\theta_1\}\right|\right|_F^2 = C\int_0^T \theta^2(0,t)dt \tag{1.20}$$

1.5 Estimate of the Least Control Time T_0

To estimate the least time to control the system (1.7) by HUM, we need to establish here the following two inequalities, which will be later used for observability of the adjoint system and hence to establish the controllability of the original problem (Dolecki and Russell [19]).

There exist positive constants C_0, C_1 and a number T_0 such that

$$\begin{aligned} C_0(T-T_0)\left[||\theta_0||^2_{L^2[0,l]} + ||\theta_1||^2_{H^{-1}[0,l]}\right] &\le C\int_0^T \theta^2(0,t)dt \\ &\le C_1T\left[||\theta_0||^2_{L^2[0,l]} + ||\theta_1||^2_{H^{-1}[0,l]}\right]. \end{aligned} \tag{1.21}$$

Before establishing the actual inequalities in (1.21), we shall first present the following two inequalities.

$$\frac{1}{2}\int_0^T \dot\theta^2(0,t)dt \ge \frac{1}{c^2\alpha + Kl}(T-T_0)E(0) \tag{1.22}$$

and

$$\frac{1}{2}\int_0^T \dot\theta^2(0,t)dt \le \frac{T}{c^2\alpha}E(0) \tag{1.23}$$

where K is some positive constant.

To establish the inequality (1.22) by multiplier technique, we multiply the first equation of (1.9) by $x\theta'$ and integrate over $[0,l]\times[0,T]$ and obtain

$$\left[\int_0^l x\theta'\dot\theta dx\right]_0^T - \frac{1}{2}\int_0^l\int_0^T x\frac{\partial}{\partial x}\dot\theta^2 dtdx = \frac{c^2}{2}\int_0^l\int_0^T x\frac{\partial}{\partial x}\theta'^2 dtdx.$$

Integrating by parts and applying the boundary conditions of (1.9), we have after a straight forward calculations

$$\left[\int_0^l x\theta'\dot{\theta}dx\right]_0^T - \frac{l}{2}\int_0^T \dot{\theta}^2(l,t)dt + \frac{1}{2}\int_0^l\int_0^T(\dot{\theta}^2 + c^2\theta'^2)dtdx = 0. \tag{1.24}$$

Now if we set

$$X = \int_0^l x\theta'\dot{\theta}dx \tag{1.25}$$

then we note that

$$|X| \le \sup_{x\in[0,l]}\{x\}\Big|\int_0^l \theta'\dot{\theta}dx\Big|. \tag{1.26}$$

Now applying the inequality

$$|ab| \le \frac{1}{2}(\epsilon a^2 + \frac{1}{\epsilon}b^2)$$

for any real $\epsilon > 0$, we have from (1.26),

$$|X| \le \frac{l}{2c}\int_0^l(\dot{\theta}^2 + c^2\theta'^2)dx = \frac{l}{c}\Big[E(t) - \frac{c^2\alpha}{2}\dot{\theta}^2(0,t)\Big] \le \frac{l}{c}E(t)$$

by the energy relation (1.10). Hence,

$$\Big|X\Big|_0^T \le \frac{l}{c}\Big(E(T) + E(0)\Big) = \frac{2l}{c}E(0) \tag{1.27}$$

by the relation (1.12).

Introducing (1.10) and (1.27) in (1.24), we have therefore

$$\begin{aligned}\frac{l}{2}\int_0^T \dot{\theta}^2(l,t)dt + \frac{c^2\alpha}{2}\int_0^T \dot{\theta}^2(0,t)dt &\ge \int_0^T E(t)dt - \frac{2l}{c}E(0)\\ &= \Big(T - \frac{2l}{c}\Big)E(0).\end{aligned} \tag{1.28}$$

Let us take a positive constant

$$K = \frac{\int_0^T \dot{\theta}^2(l,t)dt}{\int_0^T \dot{\theta}^2(0,t)dt} \tag{1.29}$$

then it follows from (1.28) that

$$\frac{1}{2}(Kl + c^2\alpha)\int_0^T \dot{\theta}^2(0,t)dt \ge (T - T_0)E(0) \tag{1.30}$$

where,

$$T_0 = \frac{2l}{c}. \tag{1.31}$$

We should note that the constant K in (1.29), gives the ratio of the total kinetic energy at the free end and that of the hub end of the panel in the time interval [0,T].

since the adjoint system is energy conserving system, therefore we can conclude that K is finite.

Next we consider the energy equation (1.10). Integrating (1.10) over $[0,T]$, we obtain

$$\int_0^T E(t)dt = \frac{1}{2}\int_0^T\int_0^l(\dot\theta^2 + c^2\theta'^2)dxdt + \frac{c^2\alpha}{2}\int_0^T \dot\theta^2(0,t)dt \geq \frac{c^2\alpha}{2}\int_0^T \dot\theta^2(0,t)dt.$$

Use of (1.12), it follows that from the above

$$\frac{c^2\alpha}{2}\int_0^T \dot\theta^2(0,t)dt \leq TE(0). \tag{1.32}$$

The relations (1.30) and (1.32) thus lead to the inequalities (1.22) and (1.23).

Now by Poincare inequality (cf. Aubin [1]), we know that the norm $(\int_0^l f'^2dx)^{\frac{1}{2}}$ is equivalent to the norm of f on the space $H^1[0,l]$, provided $f(x_0) = 0$ for $0 \leq x_0 \leq l$. Therefore there are suitable constants C_0 and C_1 such that (1.22) and (1.23) can be written as

$$\begin{aligned} C_0(T-T_0)\left[||\theta_0||^2_{H^1[0,l]} + ||\theta_1||^2_{L^2[0,l]}\right] &\leq C\int_0^T \dot\theta^2(0,t)dt \\ &\leq C_1T\left[||\theta_0||^2_{H^1[0,l]} + ||\theta_1||^2_{L^2[0,l]}\right]. \end{aligned} \tag{1.33}$$

It should be noted that the inequality of the form in (1.33) is obtained by multiplier technique (cf. [94]) which is not unique. The same form can also be obtained by different multiplier such as in the Chapter 3 and Chapter 4.

To establish the actual inequalities in (1.21) we define a premitive function $\chi(x,t)$ by the indefinite integral

$$\chi(x,t) = \int^t \theta(x,t)dt \tag{1.34}$$

so that

$$\dot\chi(x,t) = \theta(x,t).$$

Then $\chi(x,t)$ satisfies

$$\begin{aligned} \ddot\chi(x,t) - c^2\chi''(x,t) &= \dot\theta(x,t) - c^2\int^t \theta''(x,t)dt \\ &= \dot\theta(x,t) - \int^t \ddot\theta\,(x,t)dt \\ &= 0. \end{aligned}$$

Thus $\chi(x,t)$ satisfies the system of equations:

$$
\begin{aligned}
&\ddot{\chi}(x,t) = c^2\chi''(x,t), && 0 \le x \le l,\ 0 \le t \le T,\\
&\chi(x,0) = \int^{0} \theta(x,t)dt = \chi_0 \quad \text{(say)}, && 0 \le x \le l,\\
&\dot{\chi}(x,0) = \theta(x,0) = \theta_0 = \chi_1 \quad \text{(say)}, && 0 \le x \le l, \qquad (1.35)\\
&\chi'(0,t) = \int^{t} \theta'(0,t)dt = \alpha \int^{t} \ddot{\theta}(0,t)dt = \alpha\ddot{\chi}(0,t)dt, && 0 \le t \le T,\\
&\chi'(l,t) = \int^{t} \theta'(l,t)dt = 0, && 0 \le t \le T,
\end{aligned}
$$

We observe that the system (1.35) is analogous to the system (1.9). Hence we can use the inequalities in (1.33) for $\chi(x,t)$ to obtain

$$
\begin{aligned}
C_0(T-T_0)\Big[||\chi_0||^2_{H^1[0,l]} + ||\chi_1||^2_{L^2[0,l]}\Big] \ &\le\ C\int_0^T \dot{\chi}^2(0,t)dt\\
&\le\ C_1T\Big[||\chi_0||^2_{H^1[0,l]} + ||\chi_1||^2_{L^2[0,l]}\Big]. \qquad (1.36)
\end{aligned}
$$

Since $\dot{\chi}(x,t) = \theta(x,t)$, the inequalities in (1.21) are followed from (1.36).

1.6 Exact Controllability Result

In the literature, meaning of the exact controllabilty of a system is to find a suitable control function which drives whole system to a desired final state at some finite $T > 0$. Therefore to establish the exact controllability, for the solution of the system (1.7), we need an appropriate control torque $Q(t)$ defind on $[0,T]$ such that the solution of system (1.7) satisfies (1.8) and it follows from the ensuing theorem.

Theorem 1.1. Let $T > T_0$, then for every $\phi_0 \in H^1[0,l]$ and $\phi_1 \in L^2[0,l]$, there is a control function $Q(t) \in L^2[0,T]$ proportional to $\theta(0,t)$ such that $\phi(x,t)$, the solution of the system (1.7) satisfies (1.8).

Proof. For $T > T_0$, it follows from the inequalities in (1.21) that (1.19) defines a norm of $\{\theta_0,\theta_1\}$. This norm is equivalent to the norm on the Hilbert space $F = L^2[0,l] \times H^{-1}[0,l]$, being the completion of smooth functions $\{\theta_0,\theta_1\}$. Again by virtue of (1.15), (1.19) and the left inequality in (1.21), we can use Lax-Milgram theorem (cf. Aubin [1]), to conclude that Λ is an isomorphism operator from F to F' for $T > T_0$, where F' is the dual space of F. Therefore we can uniquely invert the operator Λ from F' to F. Hence, for given $\{\phi_0,\phi_1\} \in F'$, there exists $\{\theta_0,\theta_1\} \in F$ such that

$$\{\theta_0,\theta_1\} = \Lambda^{-1}\{\phi_1,\ -\phi_0\} \qquad (1.37)$$

or equivalently

$$\Lambda\{\theta_0, \theta_1\} = \{\phi_1, \ -\phi_0\}$$

Finally, if we take the control $Q(t)$ proportional to $\theta(0,t)$ say, $Q(t) = \beta_0\theta(0,t)$ for the original problem (1.7), where $\theta(x,t)$ is the solution of (1.9) with $\{\theta_0, \theta_1\}$ as solution of (1.37), then from (1.15) we have that the function $\psi(x,t)$, the solution of the system (1.14), satisfies $\psi_0 = \phi_0$ and $\psi_1 = \phi_1$. By the uniqueness of the solution, we conclude that

$$\psi(x,t) = \phi(x,t), \qquad 0 \le x \le l, \ 0 \le t \le T. \tag{1.38}$$

The result (1.8) for exact controllability of the system (1.7) then immediately follows from the backward system (1.14). This completes the proof.

1.7 Concluding Remarks

From the inequality (1.21), it follows that the adjoint system is observable for $T > T_0$, where $T_0 = 2l/c$ as given by (1.31). Therefore as in the literature [19], the the original system is controllable for $T > T_0$. The time T_0 can therefore be considered as the the estimated least time (may be thought as critical time) for exact controllability of the torsional vibration problem governed by the system (1.7). Since c is the torsional wave velocity and l the length of the panel, the least time namely, $2l/c$ can immediately be identified to the time taken by the torsional wave to originate and return to the control end (hub end) via the free end. Finally we remark that, as the action of the control torque $Q(t)$ in the system (1.7) depends on the solution of the adjoint system (1.9), the suppression of vibration at time T (exact control) entails coupling of these two systems. In this context, we mention that the adjoint system is energy conserving, while in the original system, it should decay during the control process.

CHAPTER 2

EXACT CONTROLLABILITY OF TORSIONAL VIBRATIONS OF AN INTERNALLY DAMPED FLEXIBLE PANEL*

2.1 Introduction

In the preceding Chapter, we have discussed the exact controllability of torsional vibrations of a hybrid flexible space structure in the form of a solar cell array. The mathematical formulation of the governing partial differential equation was free from any damping term. But as a correspondence of physical reality, inherent material damping of the structure, however small it may be, is always appeared in real materials (cf. Christensen [16]) as long as the system vibrates. We adopt here the simple Kelvin-Voigt model (dashpot in series with a spring) for the viscoelasticity of the structure.

In this Chapter, we like to demonstrate the exact controllability of torsional vibrations for the geomertically same hybrid structure as in Chapter 1, for a rectangular elastic panel with a rigid hub hoisted at one end and totally free at the other end. The panel is assumed to possess material damping mentioned above. Incorporation of small distributed viscous damping of the Kelvin-Voigt type appearing as an internal resistance opposing the strain velocity, makes the problem more realistic. There is basic alteration in the mathematical consideration in view of additional higher order derivative term in the governing torsional vibration equation.

2.2 Mathematical Formulation

As in the last Chapter, we consider here a simple hybrid structure consisting of a uniform rectangular elastic panel of length l held at one end by a rigid hub and the other end totally free. Our particular interest centres around the result of exact controllability of the internally damped torsional vibrations of the system, achieved with

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the help of a suitable control torque $Q(t)$ applied to the rigid hub in some finite time interval [0,T]. Internal material damping of Kelvin-Voigt type is assumed to be present as a simple model of internal damping of this panel.

Referring to the schematic Figure 1.1 in Chapter 1, if $\phi_h(t)$ is the rotation of the rigid hub and $\phi_p(x,t)$ that of the panel at the position x along the span of the panel relative to the hub at time t, then the total rotational angle $\phi(x,t) = \phi_h(t) + \phi_p(x,t)$ of the panel satisfies internally damped torsional vibration equations

$$\ddot{\phi}(x,t) = c^2\phi''(x,t) + \mu\dot{\phi}''(x,t), \qquad 0 \le x \le l, \ \ 0 \le t \le T, \tag{2.1}$$

where dots and primes denote differentiation with respect to time coordinate t and space coordinate x respectively. The constant c is the torsional wave velocity and $\mu > 0$ is the coefficient of material damping of the panel which is taken to be small enough. Initially at time $t = 0$, let the panel be set to vibrations with initial values

$$\phi(x,0) = \phi_0(x) \quad \text{and} \quad \dot{\phi}(x,0) = \phi_1(x), \qquad 0 \le x \le l. \tag{2.2}$$

In similar fashion, if at the hub end $x = 0$ the control $Q(t)$ is applied and the end $x = l$ is kept free, we obtain the same boundary conditions as (1.4)–(1.6) of Chapter 1.

Therefore, to study the exact controllability of the internally damped torsional vibrations of the uniform hybrid rectangular panel, the following initial-boundary value problem is to be concerned mathematically.

$$\begin{aligned}
&\ddot{\phi}\,(x,t) = c^2\phi''(x,t) + \mu\dot{\phi}''(x,t), && && 0 \le x \le l,\ 0 \le t \le T,\\
&\phi(x,0) = \phi_0(x), && \dot{\phi}(x,0) = \phi_1(x), && 0 \le x \le l,\\
&\phi'(0,t) = \alpha\,\ddot{\phi}\,(0,t) + \lambda Q(t), && \phi'(l,t) = 0, && 0 \le t \le T.
\end{aligned} \tag{2.3}$$

Now a control torque $Q(t)$ is to be selected appropriately, to study the exact controllability at some finite time $T > 0$, so that it drives the system (2.3) to rest at time $t = T$. Then the solution of (2.3) must satisfy

$$\phi(x,T) = \dot{\phi}(x,T) = 0. \tag{2.4}$$

To discuss the exact controllability of the mathematical problem (2.3) by HUM, some essential results are incorporated in the following sections.

2.3 Adjoint System

Associated with each solution of (2.3), we start with $\theta(x,t)$, the solution of the following adjoint system

$$\begin{aligned}
&\ddot{\theta}(x,t) = c^2\theta''(x,t) - \mu\dot{\theta}''(x,t), && 0 \le x \le l,\ 0 \le t \le T,\\
&\theta(x,0) = \theta_0(x), \qquad \dot{\theta}(x,0) = \theta_1(x), && 0 \le x \le l,\\
&\theta'(0,t) = \alpha\,\ddot{\theta}(0,t), \qquad \theta'(l,t) = 0, && 0 \le t \le T,
\end{aligned} \tag{2.5}$$

in which the governing partial differential equation is the adjoint of that in (2.3) in usual sense. In addition we assume that $\theta_0(0) = 0$, $\theta_1(0) = 0$ and $\theta_0'(0) = 0$. Now, for every $\{\theta_0, \theta_1\} \in F = L^2[0,l] \times H^{-1}[0,l]$ (where $H^{-1}[0,l]$ is the dual space of the Sobolev space $H^1[0,l]$ of order one), the system (2.5) with sufficiently small μ has a unique solution (cf. Showalter [81]). For $\mu = 0$, the system (2.5) coincides with the wave equation of pure elastic vibration having unique solution (cf. Lions and Magenes [53], Showalter [81]) for every $\{\theta_0, \theta_1\} \in F$. Existence of these smooth solutions ensure that the solution $\theta(x,t)$ tends to the elastic solution $\theta_e(x,t)$ when $\mu \to 0$. For, the difference of the solution $|\theta(x,t) - \theta_e(x,t)|$ is bounded above by a quantity proportional to μ and thus tends to zero as $\mu \to 0$.

To each solution of (2.5), the total energy at time t is defined by

$$E(t) = \frac{1}{2}\int_0^l (\dot{\theta}^2 + c^2\theta'^2)dx + \frac{c^2\alpha}{2}\dot{\theta}^2(0,t). \tag{2.6}$$

Differentiating (2.6) with respect to t and using the first equation of (2.5), it gives

$$\dot{E}(t) = c^2\int_0^l \frac{\partial}{\partial x}(\dot{\theta}\theta')dx - \mu\int_0^l \dot{\theta}\dot{\theta}''dx + c^2\alpha\dot{\theta}(0,t)\,\ddot{\theta}(0,t).$$

Integrating by parts and applying the boundary conditions of (2.5), we have from above

$$\dot{E}(t) = \mu\alpha\dot{\theta}(0,t)\,\dddot{\theta}(0,t) + \mu\int_0^l \dot{\theta}'^2dx. \tag{2.7}$$

The integral term on the right hand side of (2.7) shows that some energy is generated throughout the system which is due to the presence of some internally distributed force $(-\mu\dot{\theta}'')$ in the adjoint system (2.5). Integreating (2.7) from zero to t, we obtain the energy integral of the system as

$$E(t) = E(0) + \mu\alpha\dot{\theta}(0,t)\,\ddot{\theta}(0,t) - \mu\alpha\int_0^t \ddot{\theta}^2(0,\tau)d\tau + \mu\int_0^l\int_0^t \dot{\theta}'^2dxd\tau, \tag{2.8}$$

where

$$E(0) = \frac{1}{2}\int_0^l \left[\dot{\theta}^2(x,0) + c^2\theta'^2(x,0)\right]dx. \tag{2.9}$$

Evidenly, if $\mu = 0$, $E(t) = E(0)$ and the energy is conserved. Both the forms (2.6) and (2.8) have been used in the following.

2.4 Backward System and Operator Λ

Here we consider $\psi(x,t)$, the solution of a time backward system

$$\begin{aligned}
&\ddot{\psi}(x,t) = c^2\psi''(x,t) + \mu\dot{\psi}''(x,t), && 0 \le x \le l,\ 0 \le t \le T,\\
&\psi(x,T) = 0, \qquad \dot{\psi}(x,T) = 0, && 0 \le x \le l,\\
&\psi'(0,t) = \alpha\,\ddot{\psi}(0,t) + \lambda\beta_0\Big[\theta(0,t) + \frac{\mu}{c^2}\dot{\theta}(0,t)\Big], && 0 \le t \le T,\\
&\psi'(l,t) = 0, && 0 \le t \le T,
\end{aligned} \tag{2.10}$$

where β_0 is a constant independent of t. The solution of the nonhomogeneous boundary value problem (2.10) depends on the properties of $\theta(0,t)$ and $\dot{\theta}(0,t)$ and hence on the initial values θ_0 and θ_1 of system (2.5). Now for given $\{\theta_0,\ \theta_1\} \in F$, the system (2.10) has a solution $\psi(x,t)$ as before for the system (2.5). We can then easily obtain $\psi(x,0)$ and $\dot{\psi}(x,0)$.

Now multiplying (2.5) by ψ and (2.10) by θ, and integrating over $[0,l] \times [0,T]$ after subtraction we obtain

$$\begin{aligned}
\int_0^l\int_0^T \frac{\partial}{\partial t}(\dot{\theta}\psi - \theta\dot{\psi})dxdt &= c^2\int_0^l\int_0^T \frac{\partial}{\partial x}(\theta'\psi - \theta\psi')dxdt\\
&- \mu\int_0^l\int_0^T \Big[\frac{\partial}{\partial x}(\dot{\theta}'\psi - \dot{\theta}\psi') + \frac{\partial}{\partial t}(\theta\psi'')\Big]dxdt.
\end{aligned}$$

Applying boundary conditions of (2.5) and (2.10), it leads to

$$\begin{aligned}
\int_0^l \Big[\theta_0(\psi_1 - \mu\psi_0'') - \theta_1\psi_0\Big]dx &= \lambda\beta_0\int_0^T \Big[c^2\theta^2(0,t) - \frac{\mu^2}{c^2}\dot{\theta}^2(0,t)\Big]dt + \mu\int_0^T \frac{\partial}{\partial t}\Big[\theta'(0,t)\psi(0,t)\Big]dt\\
&+\alpha\int_0^T \Big[c^2\Big[\theta(0,t)\ddot{\psi}(0,t) - \psi(0,t)\ddot{\theta}(0,t)\Big] - \mu\Big[\dot{\theta}(0,t)\ddot{\psi}(0,t) + \dot{\psi}(0,t)\ddot{\theta}(0,t)\Big]\Big]dt
\end{aligned}$$

which after a simple calculation yields

$$\int_0^l \Big[\theta_0(\psi_1 - \mu\psi_0'') - \theta_1\psi_0\Big]dx = C\int_0^T \Big[c^2\theta^2(0,t) - \frac{\mu^2}{c^2}\dot{\theta}^2(0,t)\Big]dt \tag{2.11}$$

since $\theta_0(0) = 0$, $\theta_1(0) = 0$, $\theta_0'(0) = 0$, where $C = \lambda\beta_0$, and

$$\psi_0 = \psi(x,0), \qquad \psi_1 = \dot{\psi}(x,0). \tag{2.12}$$

At this stage we define an operator Λ as

$$\Lambda\{\theta_0,\theta_1\} = \{\psi_1 - \mu\psi_0'',\ -\psi_0\}. \tag{2.13}$$

Then from (2.11) and (2.13), the functional

$$\{\theta_0, \theta_1\} \to \Big\langle \Lambda\{\theta_0, \theta_1\}, \{\theta_0, \theta_1\}\Big\rangle$$

is given by

$$\Big\langle \Lambda\{\theta_0, \theta_1\}, \{\theta_0, \theta_1\}\Big\rangle = C\int_0^T \Big[c^2\theta^2(0,t) - \frac{\mu^2}{c^2}\dot{\theta}^2(0,t)\Big]dt. \tag{2.14}$$

In the manner of Lions [52], for T large and μ small enough, we shall verify that the functional defined by (2.14) defines a norm on the initial values $\{\theta_0, \theta_1\}$, equivalent to the norm of F, i.e.,

$$\Big|\Big|\{\theta_0, \theta_1\}\Big|\Big|_F^2 = C\int_0^T \Big[c^2\theta^2(0,t) - \frac{\mu^2}{c^2}\dot{\theta}^2(0,t)\Big]dt \tag{2.15}$$

for $T > T_0$ and $\mu < \mu_0$. T_0 and μ_0 are defined later in equations (2.34) and (2.37).

2.5 Estimate of the Least Control Time T_0

In this section we shall first prove the following Lemma.

Lemma 2.1. Let $\theta(x,t)$ be a solution of (2.5) then

$$\Big|\int_0^l\int_0^T x\dot{\theta}'\theta''dxdt\Big| \le \frac{l}{2c}\Big[TG(0) + 2\mu\int_0^l\int_0^T (T-t)\dot{\theta}''^2dxdt - \alpha\int_0^T \ddot{\theta}^2(0,t)dt\Big]$$

where

$$G(0) = \int_0^l \Big[\dot{\theta}'^2(x,0) + c^2\theta''^2(x,0)\Big]dx. \tag{2.16}$$

Proof. We have

$$\Big|\int_0^l\int_0^T x\dot{\theta}'\theta''dxdt\Big| \le \frac{l}{2c}\int_0^l\int_0^T \Big(\dot{\theta}'^2 + c^2\theta''^2\Big)dxdt. \tag{2.17}$$

Now multiplying the governing differential equation of (2.5) by $(T-t)\dot{\theta}''$ and then integrating over $[0,l]\times[0,T]$, we obtain

$$\begin{aligned}\int_0^T \Big[(T-t)\ddot{\theta}\dot{\theta}'\Big]_0^l dt \; - \; \frac{1}{2}\int_0^l\int_0^T (T-t)\frac{\partial}{\partial t}\dot{\theta}'^2dxdt \\ = \frac{c^2}{2}\int_0^l\int_0^T (T-t)\frac{\partial}{\partial t}\theta''^2dxdt - \mu\int_0^l\int_0^T (T-t)\dot{\theta}''^2dxdt.\end{aligned} \tag{2.18}$$

Using the boundary conditions of (2.5), (2.18) gives

$$\begin{aligned}\frac{1}{2}\int_0^l\int_0^T (\dot{\theta}'^2 + c^2\theta''^2)dxdt \; &= \; \frac{T}{2}\int_0^l \Big[\dot{\theta}'^2(x,0) + c^2\theta''^2(x,0)\Big]dx \\ &+\mu\int_0^l\int_0^T (T-t)\dot{\theta}''^2dxdt - \frac{\alpha}{2}\int_0^T \ddot{\theta}^2(0,t)dt.\end{aligned} \tag{2.19}$$

Hence the Lemma follows from (2.17) and (2.19).

Now we establish the following two inequalities, which will be later used for controllability of the original problem.

For $\mu < \mu_0$, there exist positive constants C_0, C_1 and a number T_0 such that

$$C_0(T-T_0)\Big[||\theta_0||^2_{L^2[0,l]} + ||\theta_1||^2_{H^{-1}[0,l]}\Big] \le C\int_0^T \Big[c^2\theta^2(0,t) - \frac{\mu^2}{c^2}\dot{\theta}^2(0,t)\Big]dt$$
$$\le C_1 T\Big[||\theta_0||^2_{L^2[0,l]} + ||\theta_1||^2_{H^{-1}[0,l]}\Big]. \tag{2.20}$$

To establish the above inequalities in (2.20), we shall need to present the following two inequalities by multiplier technique.

$$2\alpha\int_0^T \Big[c^2\dot{\theta}^2(0,t) - \frac{\mu^2}{c^2}\ddot{\theta}^2(0,t)\Big]dt \ge \frac{1-A\mu-B\mu^2}{1+K}(T-T_0)E(0) \tag{2.21}$$

and

$$\alpha\int_0^T \Big[c^2\dot{\theta}^2(0,t) - \frac{\mu^2}{c^2}\ddot{\theta}^2(0,t)\Big]dt \le 2TE(0) \tag{2.22}$$

for $0 < \mu < \mu_0$, where $K > 0$ is a real and A, B are some positive constants independent of μ.

For this, we multiply the first equation of (2.5) by $x\theta'$ and integrate by parts over $[0,l]\times[0,T]$ and obtain

$$\int_0^l \Big[x\theta'\dot{\theta}\Big]_0^T dx - \frac{1}{2}\int_0^T \Big[x\dot{\theta}^2\Big]_0^l dt + \frac{1}{2}\int_0^l\int_0^T \dot{\theta}^2 dxdt = \frac{c^2}{2}\int_0^T \Big[x\theta'^2\Big]_0^l dt$$
$$-\frac{c^2}{2}\int_0^l\int_0^T \theta'^2 dxdt + \frac{\mu}{2}\int_0^l \Big[\theta'^2\Big]_0^T dx + \mu\int_0^l\int_0^T x\dot{\theta}'\theta'' dxdt. \tag{2.23}$$

After a simple calculation (2.23) yields

$$\frac{l}{2}\int_0^T \dot{\theta}^2(l,t)dt + \frac{c^2\alpha}{2}\int_0^T \dot{\theta}^2(0,t)dt = \int_0^T E(t)dt + \int_0^l \Big[x\theta'\dot{\theta}\Big]_0^T dx$$
$$-\frac{\mu}{2}\int_0^l \Big[\theta'^2\Big]_0^T dx - \mu\int_0^l\int_0^T x\dot{\theta}'\theta'' dxdt \tag{2.24}$$

where we have used the boundary conditions of (2.5) and the energy equation (2.6).

Now if we set

$$X = \int_0^l x\theta'\dot{\theta}dx$$

then we have

$$|X| \le \frac{l}{2c}\int_0^l(\dot{\theta}^2 + c^2\theta'^2)dx = \frac{l}{c}\Big[E(t) - \frac{c^2\alpha}{2}\dot{\theta}^2(0,t)\Big].$$

Hence,

$$|X|_0^T \leq \frac{l}{c}\Big[E(T) + E(0) - \frac{c^2\alpha}{2}\dot{\theta}^2(0,T)\Big]. \tag{2.25}$$

On the other hand, we note that

$$-\frac{\mu}{c^2}E(0) \leq \frac{\mu}{2}\int_0^l \Big[\theta'^2\Big]_0^T dx \leq \frac{\mu}{c^2}\Big[E(T) - \frac{c^2\alpha}{2}\dot{\theta}^2(0,T)\Big]. \tag{2.26}$$

Introducing (2.8), (2.25) and (2.26) in (2.24), we have therefore

$$\begin{aligned}
&\frac{l}{2}\int_0^T \dot{\theta}^2(l,t)dt + \alpha\int_0^T \Big[\frac{c^2}{2}\dot{\theta}^2(0,t) - \frac{\mu^2}{c^2}\ddot{\theta}^2(0,t)\Big]dt \\
&\geq \Big[T - (\frac{2l}{c} + \frac{\mu}{c^2})\Big]E(0) + \alpha\Big[(\frac{lc}{2} + \mu)\dot{\theta}^2(0,T) - \mu(\frac{\mu}{c^2} + \frac{l}{c})\dot{\theta}(0,T)\ddot{\theta}(0,T) \\
&+\frac{\mu l}{c}\int_0^T \ddot{\theta}^2(0,t)dt - \mu\int_0^T (T-t)\ddot{\theta}^2(0,t)dt\Big] + \mu\int_0^l\int_0^T (T-t)\dot{\theta}'^2 dx dt \\
&-\mu(\frac{\mu}{c^2} + \frac{l}{c})\int_0^l\int_0^T (\dot{\theta}'^2 + c^2\theta''^2)dx dt - \mu\Big|\int_0^l\int_0^T x\dot{\theta}'\theta'' dx dt\Big|,
\end{aligned} \tag{2.27}$$

where we have used the relation

$$\int_0^T \Big(\int_0^t f(\tau)d\tau\Big)dt = \int_0^T (T-t)f(t)dt \tag{2.28}$$

for any integrable function f on $[0,T]$. This follows from change of order of integration in the double integral. Using (2.19) and Lemma 2.1, we obtain from (2.27)

$$\begin{aligned}
&\frac{l}{2}\int_0^T \dot{\theta}^2(l,t)dt + \alpha\int_0^T \Big[\frac{c^2}{2}\dot{\theta}^2(0,t) - \frac{\mu^2}{c^2}\ddot{\theta}^2(0,t)\Big]dt \\
&\geq \Big[T - (\frac{2l}{c} + \frac{\mu}{c^2})\Big]E(0) - \mu(\frac{3l}{2c} + \frac{\mu}{c^2})TG(0) + \alpha\Big[(\frac{lc}{2} + \mu)\dot{\theta}^2(0,T) \\
&-\mu(\frac{\mu}{c^2} + \frac{l}{c})\dot{\theta}(0,T)\,\ddot{\theta}\,(0,T) + \mu(\frac{5l}{2c} + \frac{\mu}{c^2})\int_0^T \ddot{\theta}^2\,(0,t)dt - \mu\int_0^T (T-t)\,\ddot{\theta}^2\,(0,t)dt\Big] \\
&+\mu\Big[\int_0^l\int_0^T (T-t)\dot{\theta}'^2 dx dt - \mu(\frac{3l}{c} + \frac{2\mu}{c^2})\int_0^l\int_0^T (T-t)\dot{\theta}''^2 dx dt\Big].
\end{aligned} \tag{2.29}$$

We note that as $\mu \to 0+$, the two integrals in the expression

$$\Big[\int_0^l\int_0^T (T-t)\dot{\theta}'^2 dx dt - \mu(\frac{3l}{c} + \frac{2\mu}{c^2})\int_0^l\int_0^T (T-t)\dot{\theta}''^2 dx dt\Big]$$

tend to corresponding integrals for the pure elastic case $\theta_e(x,t)$. Hence there exists a constant $\mu_1 > 0$ such that

$$\Big[\int_0^l\int_0^T(T-t)\dot\theta'^2dxdt-\mu(\frac{3l}{c}+\frac{2\mu}{c^2})\int_0^l\int_0^T(T-t)\dot\theta''^2dxdt\Big]\geq 0, \tag{2.30}$$

for $\mu<\mu_1$. Now we may define a positive constant K_0 such that

$$G(0)\leq K_0E(0). \tag{2.31}$$

Use of (2.30) and (2.31) in (2.29) gives

$$\begin{aligned}\frac{l}{2}\int_0^T\dot\theta^2(l,t)dt&+2\alpha\int_0^T\Big[c^2\dot\theta^2(0,t)-\frac{\mu^2}{c^2}\ddot\theta^2(0,t)\Big]dt\\ &\geq\Big[T(1-A\mu-B\mu^2)-(\frac{2l}{c}+\frac{\mu}{c^2})\Big]E(0)\\ &+\alpha\Big[(\frac{lc}{2}+\mu)\dot\theta^2(0,T)+\frac{3c^2}{2}\int_0^T\dot\theta^2(0,t)dt+\frac{5\mu l}{2c}\int_0^T\ddot\theta^2(0,t)dt\\ &-\mu\Big((\frac{\mu}{c^2}+\frac{l}{c})\dot\theta(0,T)\,\ddot\theta\,(0,T)+\int_0^T(T-t)\ddot\theta^2(0,t)dt\Big)\Big].\end{aligned} \tag{2.32}$$

By argument similar to that regarding the existence of $\mu_1>0$, the last expression within the brackets on the right hand side of (2.32) can be made nonnegative for $\mu<\mu_2$. Inequality (2.21) then follows from (2.32), where

$$K=\frac{l}{4\alpha}\frac{\int_0^T\dot\theta^2(l,t)dt}{\int_0^T\Big[c^2\dot\theta^2(0,t)-\frac{\mu^2}{c^2}\ddot\theta^2(0,t)\Big]dt}\geq\frac{l}{4\alpha}\frac{\int_0^T\dot\theta^2(l,t)dt}{\int_0^T c^2\dot\theta^2(0,t)dt}>0, \tag{2.33}$$

$$T_0=\frac{2l/c+\mu/c^2}{1-A\mu-B\mu^2} \tag{2.34}$$

and

$$A=\frac{3}{2}\frac{K_0l}{c},\qquad B=\frac{K_0}{c^2}. \tag{2.35}$$

It should be pointed out that the roots of the quadratic equation $1-A\mu-B\mu^2=0$ are real, one negative and other positive. The expression $1-A\mu-B\mu^2$ keeps the positive sign unchanged between the roots. As we have assumed that $\mu>0$ is small enough, so $1-A\mu-B\mu^2>0$ for $0<\mu<\mu_3$, when μ_3 is bounded above by the positive roots of $1-A\mu-B\mu^2=0$.

For the proof of (2.22), we consider the energy equation (2.6). Introducing (2.6) in (2.8), we obtain

$$\frac{c^2\alpha}{2}\dot{\theta}^2(0,t) = E(0) - \Big[\frac{1}{2}\int_0^l(\dot{\theta}^2 + c^2\theta'^2)dx + \mu\alpha\int_0^t \ddot{\theta}^2(0,\tau)d\tau$$
$$- \mu\alpha\dot{\theta}(0,t)\ddot{\theta}(0,t) - \mu\int_0^l\int_0^t \dot{\theta}'^2 dx d\tau\Big]. \qquad (2.36)$$

The expression in the bracket on the right hand side becomes positive as $\mu \to 0+$, hence by the same argument to that regarding the existence of $\mu_1 > 0$, there exists $\mu_4 > 0$ so that (2.36) can be written as

$$\frac{c^2\alpha}{2}\dot{\theta}^2(0,t) \leq E(0), \qquad \mu < \mu_4.$$

Hence

$$\alpha\int_0^T\Big[c^2\dot{\theta}^2(0,t) - \frac{\mu^2}{c^2}\ddot{\theta}^2(0,t)\Big]dt \leq 2TE(0).$$

Let us now define

$$\mu_0 = \min_{1\leq i\leq 4}\{\mu_i\}. \qquad (2.37)$$

Now by Poincare inequality (cf. Aubin[1]), we know that the norm $(\int_0^l f'^2 dx)^{\frac{1}{2}}$ is equivalent to the $H^1[0,l]$ norm of f, provided $f(x_0) = 0$ for $0 \leq x_0 \leq l$. Therefore from (2.21) and (2.22) we can write

$$C_0(T-T_0)\Big[||\theta_0||^2_{H^1[0,l]} + ||\theta_1||^2_{L^2[0,l]}\Big] \leq C\int_0^T\Big[c^2\dot{\theta}^2(0,t) - \frac{\mu^2}{c^2}\ddot{\theta}^2(0,t)\Big]dt$$
$$\leq C_1T\Big[||\theta_0||^2_{H^1[0,l]} + ||\theta_1||^2_{L^2[0,l]}\Big] \qquad (2.38)$$

where C_0 and C_1 are suitable positive constants.

To establish the actual inequalities in (2.20) we define a function $\chi(x,t)$ as in the preceding Chapter, by the indefinite integral

$$\chi(x,t) = \int^t \theta(x,t)dt. \qquad (2.39)$$

Then $\chi(x,t)$ satisfies

$$\begin{aligned}
\ddot{\chi}(x,t) - c^2\chi''(x,t) + \mu\dot{\chi}''(x,t) &= \dot{\theta}(x,t) - c^2\int^t \theta''(x,t)dt + \mu\theta''(x,t) \\
&= \dot{\theta}(x,t) - \int^t\Big[\ddot{\theta}(x,t) + \mu\dot{\theta}''(x,t)\Big]dt + \mu\theta''(x,t) \\
&= 0
\end{aligned}$$

that means, $\chi(x,t)$ satisfies the following system:

$$\begin{aligned}
&\ddot{\chi}(x,t) = c^2\chi''(x,t) - \mu\dot{\chi}''(x,t), && 0 \le x \le l,\ 0 \le t \le T,\\
&\chi(x,0) = \int^0 \theta(x,t)dt = \chi_0 \quad \text{(say)}, && 0 \le x \le l,\\
&\dot{\chi}(x,0) = \theta(x,0) = \theta_0 = \chi_1 \quad \text{(say)}, && 0 \le x \le l, \qquad (2.40)\\
&\chi'(0,t) = \int^t \theta'(0,t)dt = \alpha\dot{\theta}(0,t) = \alpha\ddot{\chi}(0,t), && 0 \le t \le T,\\
&\chi'(l,t) = 0, && 0 \le t \le T.
\end{aligned}$$

Therefore the system (2.40) is analogous to the system (2.5). Hence in similar fashion we can derive the inequalities in (2.38) for $\chi(x,t)$ and since $\dot{\chi}(x,t) = \theta(x,t)$, which eventually lead to the inequalities in (2.20).

2.6 Exact Controllability Result

In this section we shall prove the exact controllability theorem using HUM which is due to Lions (cf. [52]). The exact controllability for the solution of the system (2.3) follows from the following theorem:

Theorem 3.1. Let $T > T_0$ and $\mu < \mu_0$, then for every $\phi_0 \in H^2[0,l]$ and $\phi_1 \in L^2[0,l]$, there exists a control function $Q(t) \in L^2[0,T]$ such that $\phi(x,t)$, the solution of the system (2.3) satisfies (2.4).

Proof. Inequality (2.20) implies that if $T > T_0$ and $\mu < \mu_0$, (2.14) defines a norm of $\{\theta_0, \theta_1\}$ equivalent to the norm of the space F, being the completion of smooth functions $\{\theta_0, \theta_1\}$. By virtue of (2.13), (2.14) and the left inequality in (2.20), we can use Lax-Milgram theorem (cf. [1]) to conclude that Λ is an isomorphism from F to F'. Since $H^2[0,l] \subset H^1[0,l]$, there exists $\{\theta_0, \theta_1\} \in L^2[0,l] \times H^{-1}[0,l]$ such that

$$\Lambda\{\theta_0, \theta_1\} = \{\phi_1 - \mu\phi_0'', \ -\phi_0\} \tag{2.41}$$

for given $\{\phi_0, \phi_1\} \in F'$. Finally, if we take the control $Q(t) = \beta_0[\theta(0,t) + \frac{\mu}{c^2}\dot{\theta}(0,t)]$, for the original problem (2.3), where $\theta(x,t)$ is the solution of (2.5) with $\{\theta_0, \theta_1\}$ as solution of (2.41), then from (2.13) we have that the function $\psi(x,t)$, the solution of the system (2.10), satisfies $\psi_0 = \phi_0$ and $\psi_1 = \phi_1$. By the uniqueness of the solution, we conclude that

$$\psi(x,t) = \phi(x,t)$$

and the result (2.4) then follows from system (2.10).

2.7 Concluding Remarks

The inequality (2.21) implies that the adjoint system is observable for $T > T_0$ provided μ is small enough ($\mu < \mu_0$), where T_0 is given by (2.34) and μ_0 by (2.37). Therefore a minimum time of T for exact controllability of the original problem is governed by T_0 (cf. [19]). The time T_0 may be considered as the critical time which is the estimated least time for exact controllability of this internally damped vibrations of the panel. For $\mu \to 0+$, the approximated least value of T namely, $2l/c$ can immediately be identified according to the last Chapter, to the time taken by the torsional wave to originate and return to the control end. In our internally damped torsional vibrations problem, the estimated least time T_0 is slightly (because of smallness of μ) greater than $2l/c$, which is due to the presence of a small distributed force $(-\mu\dot{\theta}'')$ in the adjoint system (2.5). In other words, it can be conceived that the small internal damping resists to propagate the control wave through the viscoelastic panel and as a result the estimated least control time for this case is slightly larger than that of undamped system of torsional vibration.

CHAPTER 3

EXACT CONTROLLABILITY OF FLEXURAL VIBRATIONS OF A FLEXIBLE PANEL*

3.1 Introduction

In the earlier two Chapters, we have concerned about the undamped torsional and internally damped torsional vibrations of a flexible hybrid structure consisting of uniform rectangular elastic panel and a rigid hub at one end. But among the various type of vibrations, the transverse vibration is the most common for vibrations of beam or plate, appearing in the literature. In this Chapter, we study the exact controllability of flexural vibrations of the same geometrical flexible space structure as in the previous Chapters. The dynamics of the vibrations is mathematically governed by one dimensional fourth order Petrowsky equation or Euler-Bernoulli beam equation. The rigid hub affixed at one end of the panel is assumed to be capable of motion in the transverse direction. Installation of the movable hub at one end of the panel leads to a non-standard hybrid system. An active control force is applied only on the hub to supress the vibrations of the system exactly when the motion is set from given initial displacement and velocity along the length of the panel. Also an estimate of the minimum time of control is obtained theoretically by HUM.

Due to building of larger and more flexible space structures, designing and vibration control of these have received a great deal of attention. The vibration control of flexible space structures is very signficant from both mathematical and engineering points of view, belonging to the distributed parameter control problem. In engineering literature, the convenient and practical approach manifested into these problems is to decompose the vibrations into normal modes and consider only first few modes to reduce the problem to a finite dimensional state space representation (cf. Bontsema *et al.* [3], Fukuda *et al.* [22,23,24], Matsuno *et al.* [59]) while, in mathematical literature, the same is treated as distributed system governed by partial differential equation (cf. Lions [52], Lasiecka and Triggiani [47,48]).

*The contents of the chapter have been published in the paper *Exact Controllability of a Linear Euler-Bernoulli Panel* —Gorain and Bose, 'Journal of Sound of Vibration', Vol. 217, 637–652, (1998).

The question of controllability and stability for the vibration of the Euler-Bernoulli beam clamped at one end, with boundary control at the other end has been studied theoretically by Littman and Markus [54,55]. The idea is extended by Markus and You [58] to obtain an approximate control system. The problem of controllability and stabil ity for serially connected beams with actuators and sensors co-located at nodal points has been discussed by Chen *et al.* [12]. Morgül [66] treated the case of controllability of Euler-Bernoulli beam using the energy functional of the system while Nagaya [68] looked forward to cancel resonances subject to forced vibrations applying inertia force cancellation method. The controllability of hybrid elastic system governed by Euler-Bernoulli beam equation clamped at one end, with boundary control at the free end has been discussed by Littaman and Markus [54], and the result of it later generalised by Rao [73].

3.2 Mathematical Formulation

As a simple model, we consider a uniform rectangular flexible panel hoisted by a rigid hub at one side as shown in Figure 3.1. The panel of length l, unit width, having uniform mass density m per unit length, is rigidly attached with the hub of mass m_h at one end and totally free at the other end. The attached end can be thought as lumped mass capable of lateral motion under active control force. Our aim is to control the vibrations of the panel exactly by applying suitable control force $Q(t)$ on the rigid hub, in some finite time interval [0,T], when it is initially set in motion. If $y_h(t)$ be the transverse displacement of the rigid hub and $y_p(x,t)$ that of the panel at the position x along the span of the panel relative to the hub at time t, then the total transverse deflection can be written as

$$y(x,t) = y_h(t) + y_p(x,t), \qquad 0 \le x \le l,\ 0 \le t \le T. \tag{3.1}$$

Let us assume that the vibrations undergo only small deformations, i.e., $|y(x,t)| << l$ and $|(\partial y/\partial x)(x,t)| << 1$, and neglect the gravitational effect and rotatory inertia of the panel cross-sections. Then $y(x,t)$ satisfies the one dimensional fourth order Petrowsky equation

$$m\frac{\partial^2 y}{\partial t^2}(x,t) + D\frac{\partial^4 y}{\partial x^4}(x,t) = 0, \qquad 0 \le x \le l,\ 0 \le t \le T, \tag{3.2}$$

where $D = (1/12)Eh^3(1-\nu^2)^{-1}$. The constants D, E, ν and h are the flexural rigidity, the Young's modulus, the Poisson's ratio and the thickness of the panel respectively.

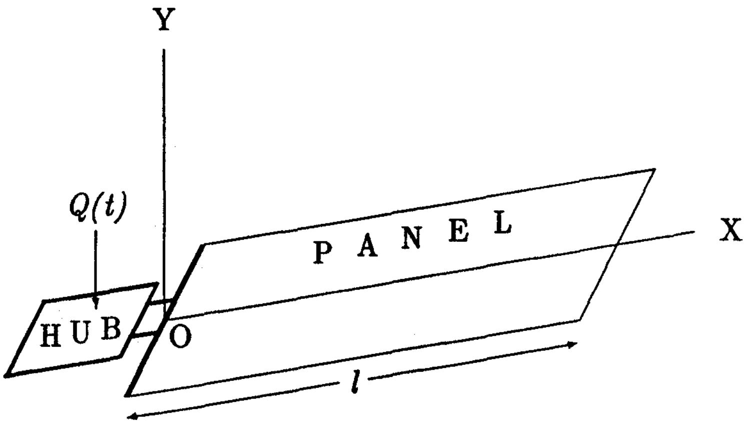

Figure 3.1. *Schematic of the rigid hub and the panel for flexural vibrations.*

At the hub end $x = 0$ where the control force $Q(t)$ is applied, the hub dynamics yields the differential equation

$$m_h \frac{\partial^2 y_h}{\partial t^2}(t) + D\frac{\partial^3 y_p}{\partial x^3}(0,t) + Q(t) = 0. \tag{3.3}$$

Again since $y_p(0,t) = 0$, it follows from (3.1) that $y(0,t) = y_h(t)$ and also we have $(\partial y/\partial x)(x,t) = (\partial y_p/\partial x)(x,t)$. Hence the equation (3.3) becomes

$$\frac{\partial^3 y}{\partial x^3}(0,t) + \alpha\frac{\partial^2 y}{\partial t^2}(0,t) + \lambda Q(t) = 0, \qquad 0 \le t \le T, \tag{3.4}$$

where $\alpha = m_h/D$ and $\lambda = 1/D$. Assuming at $x = 0$, that there is no rotational deflection of the panel relative to the hub (i.e., panel is built-in position with hub at $x = 0$), we have $(\partial y_p/\partial x)(0,t) = 0$, implying

$$\frac{\partial y}{\partial x}(0,t) = 0, \qquad 0 \le t \le T. \tag{3.5}$$

Since the panel is assumed to be free at $x = l$, so at this end

$$\frac{\partial^2 y}{\partial x^2}(l,t) = 0 \quad \text{and} \quad \frac{\partial^3 y}{\partial x^3}(l,t) = 0, \qquad 0 \le t \le T. \tag{3.6}$$

Let the panel be set to vibrations with arbitrary initial values

$$y(x,0) = y_0(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = y_1(x), \qquad 0 \le x \le l. \tag{3.7}$$

Therefore, the mathematical model to be studied for exact controllability of flexural vibrations of a uniform rectangular flexible panel with a rigid hub at one end, is governed by the system of equations:

$$
\begin{aligned}
& m\frac{\partial^2 y}{\partial t^2}(x,t) + D\frac{\partial^4 y}{\partial x^4}(x,t) = 0, && 0 \le x \le l,\ 0 \le t \le T, \\
& \frac{\partial^3 y}{\partial x^3}(0,t) + \alpha\frac{\partial^2 y}{\partial t^2}(0,t) + \lambda Q(t) = 0, \quad \frac{\partial y}{\partial x}(0,t) = 0, && 0 \le t \le T, \\
& \frac{\partial^2 y}{\partial x^2}(l,t) = 0, \quad \frac{\partial^3 y}{\partial x^3}(l,t) = 0, && 0 \le t \le T, \\
& y(x,0) = y_0(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = y_1(x), && 0 \le x \le l.
\end{aligned}
\tag{3.8}
$$

3.3 Adjoint System

Associated with each solution of (3.8), we consider with $\theta(x,t)$, the solution of its adjoint system:

$$
\begin{aligned}
& m\frac{\partial^2 \theta}{\partial t^2}(x,t) + D\frac{\partial^4 \theta}{\partial x^4}(x,t) = 0, && 0 \le x \le l,\ 0 \le t \le T, \\
& \frac{\partial^3 \theta}{\partial x^3}(0,t) + \alpha\frac{\partial^2 \theta}{\partial t^2}(0,t) = 0, \quad \frac{\partial \theta}{\partial x}(0,t) = 0, && 0 \le t \le T, \\
& \frac{\partial^2 \theta}{\partial x^2}(l,t) = 0, \quad \frac{\partial^3 \theta}{\partial x^3}(l,t) = 0, && 0 \le t \le T, \\
& \theta(x,0) = \theta_0(x) \quad \text{and} \quad \frac{\partial \theta}{\partial t}(x,0) = \theta_1(x), && 0 \le x \le l,
\end{aligned}
\tag{3.9}
$$

under the assumptions $\theta_0(0) = 0$ and $\theta_1(0) = 0$. Now for given $\{\theta_0,\ \theta_1\}$ in the space $F = L^2[0,l] \times H^{-2}[0,l]$, the system (3.9) has a unique solution $\theta(x,t)$ (cf. Lions and Magenes [53]) for $0 \le x \le l$, $0 \le t \le T$, where $H^{-2}[0,l]$ is the dual space of $H^2[0,l]$ and $H^2[0,l]$ the sobolev space of order 2, given by

$$H^2[0,l] = \left\{f \mid f \in L^2[0,l],\ \frac{\partial f}{\partial x} \in L^2[0,l],\ \frac{\partial^2 f}{\partial x^2} \in L^2[0,l]\right\}.$$

To every solution of (3.9), the total energy $E(t)$ at time t is defined by

$$E(t) = \frac{1}{2}\int_0^l \left[m\Big(\frac{\partial \theta}{\partial t}\Big)^2 + D\Big(\frac{\partial^2 \theta}{\partial x^2}\Big)^2\right]dx + \frac{1}{2}m_h\Big[\frac{\partial \theta}{\partial t}(0,t)\Big]^2. \tag{3.10}$$

Differentiating with respect to t and replacing $m(\partial^2\theta/\partial t^2)$ by $-D(\partial^4\theta/\partial x^4)$, we obtain from (3.10)

$$\frac{dE}{dt} = D\int_0^l \frac{\partial}{\partial x}\Big(\frac{\partial^2\theta}{\partial t\partial x}\frac{\partial^2\theta}{\partial x^2} - \frac{\partial\theta}{\partial t}\frac{\partial^3\theta}{\partial x^3}\Big)dx + m_h\frac{\partial\theta}{\partial t}(0,t)\frac{\partial^2\theta}{\partial t^2}(0,t).$$

Integrating by parts and applying the boundary conditions of (3.9), the above after a simplification gives

$$\frac{dE}{dt} = 0$$

since $\alpha = m_h/D$, which implies

$$E(t) = \text{constant} = E(0) \qquad \text{for} \quad 0 \le t \le T, \tag{3.11}$$

where

$$E(0) = \frac{1}{2}\int_0^l \Big[m\Big(\frac{\partial\theta}{\partial t}(x,0)\Big)^2 + D\Big(\frac{\partial^2\theta}{\partial x^2}(x,0)\Big)^2\Big]dx, \tag{3.12}$$

since $\theta_1(0) = 0$. Thus the adjoint system is energy conserving.

3.4 Backward System and Operator Λ

On the other hand, marching backward in time, we now consider a system :

$$\begin{aligned}
&m\frac{\partial^2\phi}{\partial t^2}(x,t) + D\frac{\partial^4\phi}{\partial x^4}(x,t) = 0, && 0\le x\le l,\ 0\le t\le T,\\
&\frac{\partial^3\phi}{\partial x^3}(0,t) + \alpha\frac{\partial^2\phi}{\partial t^2}(0,t) + \beta_0\lambda\theta(0,t) = 0, \quad \frac{\partial\phi}{\partial x}(0,t) = 0, && 0\le t\le T,\\
&\frac{\partial^2\phi}{\partial x^2}(l,t) = 0, \qquad \frac{\partial^3\phi}{\partial x^3}(l,t) = 0, && 0\le t\le T,\\
&\phi(x,T) = 0 \quad \text{and} \quad \frac{\partial\phi}{\partial t}(x,T) = 0, && 0\le x\le l,
\end{aligned} \tag{3.13}$$

where, β_0 is a constant independent of t. From the nonhomogeneous boundary value problem (3.13), it is clear that solution $\phi(x,t)$ depends on the initial values $\{\theta_0,\ \theta_1\}$ of (3.9), since $\theta(0,t)$ explicitly occurs in the boundary condition of (3.13). Now for a given $\{\theta_0,\ \theta_1\} \in F$, the system (3.13) has a solution $\phi(x,t)$ (cf. Lions and Magenes [53]). Thus knowing $\phi(x,t)$, the functions $\phi(x,0)$ and $(\partial\phi/\partial t)(x,0)$ can easily be obtained. Therefore we can define an operator Λ uniquely as

$$\Lambda\{\theta_0,\ \theta_1\} = \{\phi_1,\ -\phi_0\}, \tag{3.14}$$

where $\phi_0 = \phi(x,0)$ and $\phi_1 = (\partial\phi/\partial t)(x,0)$.

We shall now estimate the functional

$$\{\theta_0,\theta_1\} \rightarrow \left\langle \Lambda\{\theta_0,\theta_1\},\{\theta_0,\theta_1\}\right\rangle$$

given by

$$\left\langle \Lambda\{\theta_0,\theta_1\},\{\theta_0,\theta_1\}\right\rangle = \int_0^l (\theta_0\phi_1-\theta_1\phi_0)dx. \tag{3.15}$$

For this, we multiply the first equation of (3.9) by ϕ and that of (3.13) by θ, integrate over $[0,l]\times[0,T]$ and then subtract :

$$m\int_0^l\int_0^T \frac{\partial}{\partial t}\Big(\frac{\partial\theta}{\partial t}\phi-\theta\frac{\partial\phi}{\partial t}\Big)dxdt + D\int_0^l\int_0^T \frac{\partial}{\partial x}\Big(\frac{\partial^3\theta}{\partial x^3}\phi-\theta\frac{\partial^3\phi}{\partial x^3}\Big)dxdt$$
$$-D\int_0^l\int_0^T \frac{\partial}{\partial x}\Big(\frac{\partial^2\theta}{\partial x^2}\frac{\partial\phi}{\partial x}-\frac{\partial\theta}{\partial x}\frac{\partial^2\phi}{\partial x^2}\Big)dxdt = 0.$$

Using the boundary, initial and final conditions of two systems (3.9) and (3.13), a straightforward calculation gives

$$\begin{aligned} m\int_0^l(\theta_0\phi_1-\theta_1\phi_0)dx &= \beta_0\int_0^T \theta^2(0,t)dt \\ &\quad -D\alpha\int_0^T \Big[\frac{\partial^2\theta}{\partial t^2}(0,t)\phi(0,t)-\theta(0,t)\frac{\partial^2\phi}{\partial t^2}(0,t)\Big]dt \end{aligned}$$

which after simplification finally yields

$$\int_0^l(\theta_0\phi_1-\theta_1\phi_0)dx = C\int_0^T \theta^2(0,t)dt \tag{3.16}$$

by the systems (3.9) and (3.13), where $C=\beta_0/m$. Thus the functional (3.15) is obtained as

$$\left\langle \Lambda\{\theta_0,\theta_1\},\{\theta_0,\theta_1\}\right\rangle = \int_0^l(\theta_0\phi_1-\theta_1\phi_0)dx = C\int_0^T \theta^2(0,t)dt. \tag{3.17}$$

Following Lions [52], we shall verify that for $T>T_0$ (T_0 is estimated in the next section), the right hand side of (3.17) defines a norm on the initial values $\{\theta_0,\theta_1\}$, equivalent to the norm on the space F, i.e.,

$$\left\|\{\theta_0,\theta_1\}\right\|_F^2 = C\int_0^T \theta^2(0,t)dt. \tag{3.18}$$

3.5 Estimate of the Least Control Time T_0

To estimate the least time to control the system (3.8) by HUM, we require to establish the following two inequalities.

$$TE(0) \geq \frac{m_h}{2}\int_0^T \Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2 dt \geq \frac{T-T_0}{1+Kl}E(0) \tag{3.19}$$

for some positive constant K.

For this, we now multiply the first equation of (3.9) by $(l-x)(\partial\theta/\partial x)$ and integrate by parts over $[0,l]\times[0,T]$, and use of the boundary conditions of (3.9) to obtain

$$\left[m\int_0^l (l-x)\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial t}dx\right]_0^T - \frac{1}{2}\int_0^l\int_0^T (l-x)\frac{\partial}{\partial x}\left[m\left(\frac{\partial\theta}{\partial t}\right)^2 + D\left(\frac{\partial^2\theta}{\partial x^2}\right)^2\right]dxdt$$

$$+D\int_O^l\int_0^T \frac{\partial\theta}{\partial x}\frac{\partial^3\theta}{\partial x^3}dxdt = 0.$$

Again integrating by parts and simplifying, the above leads to

$$\frac{l}{2}\int_0^T\left[m\left(\frac{\partial\theta}{\partial t}(0,t)\right)^2 + D\left(\frac{\partial^2\theta}{\partial x^2}(0,t)\right)^2\right]dt + \left[m\int_0^l (l-x)\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial t}dx\right]_0^T$$

$$\geq \frac{1}{2}\int_0^l\int_0^T\left[m\left(\frac{\partial\theta}{\partial t}\right)^2 + D\left(\frac{\partial^2\theta}{\partial x^2}\right)^2\right]dt. \tag{3.20}$$

Since $(\partial\theta/\partial x)(0,t)=0$, from Wirtinger's inequality (cf. [80]), we can write

$$\int_0^l\left(\frac{\partial\theta}{\partial x}\right)^2 dx \leq \frac{4l^2}{\pi^2}\int_0^l\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx. \tag{3.21}$$

On the other hand, if we set

$$X = m\int_0^l (l-x)\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial t}dx \tag{3.22}$$

then using the inequality

$$|ab| \leq \frac{1}{2}\left(\epsilon a^2 + \frac{1}{\epsilon}b^2\right) \tag{3.23}$$

for any real positive ϵ, we can majorize (3.22) as

$$|X| \leq m\int_0^l |l-x|\left|\frac{\partial\theta}{\partial x}\right|\left|\frac{\partial\theta}{\partial t}\right|dx \leq \frac{l^2}{\pi}\sqrt{\frac{m}{D}}\int_0^l\left[m\left(\frac{\partial\theta}{\partial t}\right)^2 + D\frac{\pi^2}{4l^2}\left(\frac{\partial\theta}{\partial x}\right)^2\right]dx$$

$$\leq \frac{2l^2}{\pi}\sqrt{\frac{m}{D}}E(t)$$

by inequality (3.21) and energy relation (3.10). Hence,

$$|X|_0^T \leq \frac{2l^2}{\pi}\sqrt{\frac{m}{D}}\left[E(T)+E(0)\right] = \frac{4l^2}{\pi}\sqrt{\frac{m}{D}}E(0) \tag{3.24}$$

in virtue of (3.11).

Introducing (3.10) and (3.24) into (3.20), we have therefore

$$\begin{aligned}\frac{m_h}{2}\int_0^T\Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2dt \;&+\; \frac{l}{2}\int_0^T\Big[m\Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2+D\Big(\frac{\partial^2\theta}{\partial x^2}(0,t)\Big)^2\Big]dt\\ &\geq \int_0^T E(t)dt-\frac{4l^2}{\pi}\sqrt{\frac{m}{D}}E(0)\\ &= \Big(T-\frac{4l^2}{\pi}\sqrt{\frac{m}{D}}\Big)E(0). \qquad (3.25)\end{aligned}$$

Again if we set the positive constants K and T_0 by

$$K=\frac{\int_0^T\Big[\big(m\frac{\partial\theta}{\partial t}(0,t)\big)^2+D\big(\frac{\partial^2\theta}{\partial x^2}(0,t)\big)^2\Big]dt}{m_h\int_0^T\big(\frac{\partial\theta}{\partial t}(0,t)\big)^2dt}>0 \qquad (3.26)$$

and

$$T_0=\frac{4l^2}{\pi}\sqrt{\frac{m}{D}}, \qquad (3.27)$$

then it follows from (3.25) that

$$\frac{m_h}{2}(1+Kl)\int_0^T\Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2dt\geq(T-T_0)E(0). \qquad (3.28)$$

Physically the value of K is the ratio of the total kinetic and bending energy at the joint of the panel with the hub, and that of kinetic energy of the hub. Since the energy of the system is conserved, K is bounded above for all T.

Again integrating the energy relation (3.10) over $[0,T]$, we can write easily

$$\frac{m_h}{2}\int_0^T\Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2dt\leq TE(0) \qquad (3.29)$$

with the help of (3.11). Hence we have the result (3.19) from (3.28) and (3.29).

We now define a premitive function $\psi(x,t)$ satisfying the indefinite integral

$$\psi(x,t)=\int^t\theta(x,t)dt. \qquad (3.30)$$

Then $\psi(x,t)$ satifies

$$\begin{aligned}m\frac{\partial^2\psi}{\partial t^2}(x,t)+D\frac{\partial^4\psi}{\partial x^4}(x,t) &= m\frac{\partial\theta}{\partial t}(x,t)+D\int^t\frac{\partial^4\theta}{\partial x^4}(x,t)dt\\ &= m\frac{\partial\theta}{\partial t}(x,t)-\int^t m\frac{\partial^2\theta}{\partial t^2}dt\\ &= 0.\end{aligned}$$

Thus $\psi(x,t)$ satisfies the system:

$$
\begin{aligned}
& m\frac{\partial^2\psi}{\partial t^2}(x,t) + D\frac{\partial^4\psi}{\partial x^4}(x,t) = 0, && 0 \le x \le l,\ 0 \le t \le T,\\
& \frac{\partial^3\psi}{\partial x^3}(0,t) + \alpha\frac{\partial^2\psi}{\partial t^2}(0,t) = 0, \qquad \frac{\partial\psi}{\partial x}(0,t) = 0, && 0 \le t \le T,\\
& \frac{\partial^2\psi}{\partial x^2}(l,t) = 0, \qquad \frac{\partial^3\psi}{\partial x^3}(l,t) = 0, && 0 \le t \le T,\\
& \psi(x,0) = \int^t \theta(x,0)dt\Big|_{t=0} = \psi_0(x) \quad \text{say}, && 0 \le x \le l,\\
& \frac{\partial\psi}{\partial t}(x,0) = \theta(x,0) = \theta_0 = \psi_1(x), \quad \text{say}, && 0 \le x \le l.
\end{aligned}
\tag{3.31}
$$

We observe that the system (3.31) is analogous to the system (3.9). Hence the inequalities in (3.19) can be used for $\psi(x,t)$. Since $(\partial\psi/\partial t)(x,t) = \theta(x,t)$, using the inequalities in (3.19) for the solution of the system (3.31), we have

$$
\begin{aligned}
\frac{T}{2}\int_0^l \Big[m\Big(\frac{\partial\psi}{\partial t}(x,0)\Big)^2 + D\Big(\frac{\partial^2\psi}{\partial x^2}(x,0)\Big)^2\Big]dx &\ge \frac{m_h}{2}\int_0^T \Big(\frac{\partial\psi}{\partial t}(0,t)\Big)^2 dt\\
&\ge \frac{T-T_0}{2(1+Kl)}\int_0^l \Big[m\Big(\frac{\partial\psi}{\partial t}(x,0)\Big)^2 + D\Big(\frac{\partial^2\psi}{\partial x^2}(x,0)\Big)^2\Big]dx.
\end{aligned}
\tag{3.32}
$$

By Poincare inequality (cf. Aubin [1]), it follows that the norm

$$
\Big|\Big|\frac{\partial^2\psi}{\partial x^2}\Big|\Big|^2_{L^2[0,l]} = \int_0^l \Big(\frac{\partial^2\psi}{\partial x^2}\Big)^2 dx
$$

is equivalent to the norm of ψ in $H^2[0,l]$. Therefore, there exists positive numbers C_0 and C_1 such that (3.32) can be written as

$$
\begin{aligned}
C_1T\Big[||\psi_1||^2_{L^2[0,l]} + ||\psi_0||^2_{H^2[0,l]}\Big] &\ge C\int_0^T \Big(\frac{\partial\psi}{\partial t}(0,t)\Big)dt\\
&\ge C_0(T-T_0)\Big[||\psi_1||^2_{L^2[0,l]} + ||\psi_0||^2_{H^2[0,l]}\Big].
\end{aligned}
\tag{3.33}
$$

Thus the above eventually yields

$$
\begin{aligned}
C_1T\Big[||\theta_0||^2_{L^2[0,l]} + ||\theta_1||^2_{H^{-2}[0,l]}\Big] &\ge C\int_0^T \big(\theta(0,t)\big)^2 dt\\
&\ge C_0(T-T_0)\Big[||\theta_0||^2_{L^2[0,l]} + ||\theta_1||^2_{H^{-2}[0,l]}\Big],
\end{aligned}
\tag{3.34}
$$

in virtue of (3.30). The inequalities in (3.34) will help to estalish the exact controllability result by HUM, in the following section.

3.6 Exact Controllability Result

To study the exact controllability at some finite time $T > 0$, we require to select an appropriate $Q(t)$ for the system (3.8) such that it drives the system to rest at time $t = T$. Then the solution of the system (3.8) must satisfy the desired final state

$$y(x,T) = 0 \quad \text{and} \quad \frac{\partial y}{\partial t}(x,T) = 0, \quad 0 \le x \le l, \tag{3.35}$$

and this follows from the ensueing theorem.

Theorem 3.1. Let $T > T_0$, then for every $y_0 \in H^2[0,l]$ and $y_1 \in L^2[0,l]$, there is a control function $Q(t) \in L^2[0,T]$ such that $y(x,t)$, the solution of the system (3.8) satisfies (3.35).

Proof. From the inequalities in (3.34), we can conclude that for $T > T_0$, (3.17) defines a norm of $\{\theta_0, \theta_1\}$, which is equivalent to the norm on the Hilbert space $F = L^2[0,l] \times H^{-2}[0,l]$. From the right inequality of (3.34), we can invoke the Lax-Milgram theorem (cf. Aubin [1]) in virtue of (3.14) and (3.17) to conclude that Λ is an isomorphism from F to F' where, F' is the dual space of F. Hence for given $\{y_0, y_1\} \in F'$, there exists $\{\theta_0, \theta_1\} \in F$ from the relation

$$\{\theta_0, \theta_1\} = \Lambda^{-1}\{y_1, -y_0\}. \tag{3.36}$$

Now if we take the control $Q(t)$ in (3.8) of the original problem as proportional to $\theta(0,t)$ say, $Q(t) = \beta_0\theta(0,t)$, where $\theta(x,t)$ is the solution of (3.9) with $\{\theta_0, \theta_1\}$ as solution of (3.36), then from (3.14), we have that the function $\phi(x,t)$, the solution of the system (3.13), satisfies $\phi_0 = y_0$ and $\phi_1 = y_1$. By the uniqueness theorem we finally conclude that

$$\phi(x,t) = y(x,t), \quad 0 \le x \le l,\ 0 \le t \le T, \tag{3.37}$$

and the result (3.35) for exact controllability of the original system (3.8) then follows from the system (3.13). Hence the theorem.

3.7 Concluding Remarks

In mathematical literature, (3.34) provides an observability result for positivity of right hand side of it. Thus the adjoint system is observable for $T > T_0$. Hence the vibrations of the original problem can be exactly controlled for $T > T_0$ by Dolecki and Russell [19], where T_0 is given by (3.27). Therefore the time T_0 can be described as the estimated least time for exact controllability of this system. Though we have established the results for transerve vibrations of a flexible panel attached to a rigid hub at one end, in this context it should be mentioned that, the results are analogously also valid for an Euler-Bernoulli beam held by a rigid hub at one end. In this case EI, the flexural rigidity of the beam takes the place that of D of the panel.

An interpretation of T_0 is as follows. The frequency of vibrations of a uniform panel (or bar) fixed (clamped) at $x = 0$ and free at $x = l$ is $(1/2\pi l^2)\sqrt{D/m}\, p^2$, where p is a root of the equation (cf. Clough and Penzien [17])

$$\cos p \,\cosh p + 1 = 0. \tag{3.38}$$

The roots of (3.38) are approximately given by $p_1 = 1.875$, $p_2 = 4.694$ etc. If τ be the time period of the first (gravest) mode of vibration then

$$\tau = \frac{2\pi l^2}{p_1^2}\sqrt{\frac{m}{D}}. \tag{3.39}$$

Therefore

$$\frac{T_0}{\tau} = \frac{2p_1^2}{\pi^2} = 0.71 \quad \text{(approximately)}. \tag{3.40}$$

Hence T_0 is somewhat less than τ. The deflation in time period may be ascribed to the compliant motion of the end $x = 0$ towards the equlibrium position $y = 0$.

CHAPTER 4

EXACT CONTROLLABILITY OF FLEXURAL VIBRATIONS OF AN INTERNALLY DAMPED FLEXIBLE PANEL*

4.1 Introduction

In this Chapter, we want to extend the result of exact controllability for the flexural vibration problem of the Chapter 3, by incorporation of the small internal damping (Kelvin-Voigt model of viscoelasticity) of the material. This distributed small viscous damping of internal resistance opposing the strain velocity during the process of vibration, makes the dynamics of the vibration more realistic in form. The motivation of incorporating internal damping of the elastic structure arises due to the fact that such an effect however small it may be, always appears in the dynamics of physical systems (cf. Christensen [16]). There is significant difference from mathematical point of view due to incorporation of internal material damping of the viscoelastic structure, even though it it a similar geometrical problem, modeled by a rectangular flexible panel, attached to a rigid hub at one end on which an active control force is applicable and totally free at the other end, as in the last Chapter. With this manifesto, using HUM, it is to be established theoretically the exact controllability of the hybrid system in as much as short time, to achieve the desired null controllability result.

4.2 Mathematical Formulation

We consider here the same hybrid structure, consisting of a uniform rectangular elastic panel, having of length l, unit width, uniform mass density m per unit length and a rigid hub hoisted at one side of it with totally free at the other as shown in Figure 3.1, in the previous Chapter. Let us assume that small internal material damping, uniform of constant measure $\mu > 0$ (in the Kelvin-Voigt type) is present for

*The contents of this chapter have been commmunicated in the form of a paper *Exact controllability and Boundary Stabilization of Flexural Vibrations of an Internally Damped Flexible Space Structure* — Gorain and Bose, 'Applied Mathematics and Computation'.

the panel material. Here the objective is to control the vibration of the system exactly i.e., to achieve the null controllability by application of an appropriate active control force $Q(t)$ on the rigid hub, in some time interval $[0,T]$ when it is initially set in motion. Referring to the schematic Figure 3.1, if $y_h(t)$ be the transverse displacement of the rigid hub from some equilibrium position and $y_p(x,t)$ that of the panel at the position x along the span of the panel relative to the hub, at time t, then the total transverse deflection $y(x,t) = y_h(t)+y_p(x,t)$ satisfies the Voigt-type internally damped transverse vibration equation (cf. Clough and Penzien [17])

$$m\frac{\partial^2 y}{\partial t^2}(x,t) + \mu D\frac{\partial^5 y}{\partial x^4 \partial t}(x,t) + D\frac{\partial^4 y}{\partial x^4}(x,t) = 0, \qquad 0 \le x \le l,\ 0 \le t \le T, \tag{4.1}$$

under the assumption that the vibrations undergo only small deformations, that means $|y(x,t)| << l$ and $|(\partial y/\partial x)(x,t)| << 1$ and neglecting the gravitational effect and rotatory inertia of the panel cross-sections. We further assume that the effect of the term $\mu D(\partial^5 y/\partial x^4 \partial t)$ in (4.1), appearing due to consideration of Kelvin-Voigt model of viscoelasticity, is very small compared to that of the pure elastic term $D(\partial^4 y/\partial x^4)$ in (4.1).

Let us consider that the material of the panel is isotropic and the internal damping constant for the flexural vibration of the panel is same as that of the panel at the hub end. Therefore, at the hub end $x = 0$ where the control force $Q(t)$ is applied, the equation of motion is

$$m_h\frac{\partial^2 y_h}{\partial t^2}(t) + D\frac{\partial^3 y_p}{\partial x^3}(0,t) + \mu D\frac{\partial^4 y_p}{\partial x^3 \partial t}(0,t) + Q(t) = 0, \qquad 0 \le t \le T. \tag{4.2}$$

Taking into account the internal material damping of the isotropic panel, the above equation is more appropriate but a little bit different from that of (3.3) in Chapter 3. As $y(0,t) = y_h(t)$ and $(\partial y/\partial x)(x,t) = (\partial y_p/\partial x)(x,t)$, the above equation thus leads to

$$\frac{\partial^3 y}{\partial x^3}(0,t) + \mu\frac{\partial^4 y}{\partial x^3 \partial t}(0,t) + \alpha\frac{\partial^2 y}{\partial t^2}(0,t) + \lambda Q(t) = 0, \qquad 0 \le t \le T, \tag{4.3}$$

where $\alpha = m_h/D$ and $\lambda = 1/D$. As in the case of equation (4.1), the viscoelatic second term in the above equation (4.3), is assumed much smaller than the pure elastic first term. Considering that the panel is built-in position with hub at $x = 0$, i.e., there is no rotational deflection of the panel relative to the hub, we have

$$\frac{\partial y}{\partial x}(0,t) = 0, \qquad 0 \le t \le T. \tag{4.4}$$

The free end $x = l$ of the panel yields the conditions

$$\frac{\partial^2 y}{\partial x^2}(l,t) = 0 \quad \text{and} \quad \frac{\partial^3 y}{\partial x^3}(l,t) = 0, \qquad 0 \le t \le T. \tag{4.5}$$

Initially the panel is set to vibrations with

$$y(x,0) = y_0(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = y_1(x), \qquad 0 \le x \le l. \tag{4.6}$$

Therefore, the mathematical problem to be studied for exact controllability for the vibrations of internally damped hybrid panel is governed by the following system.

$$\begin{aligned}
& m\frac{\partial^2 y}{\partial t^2}(x,t) + \mu D\frac{\partial^5 y}{\partial x^4 \partial t}(x,t) + D\frac{\partial^4 y}{\partial x^4}(x,t) = 0, && 0 \le x \le l,\ 0 \le t \le T, \\
& \frac{\partial^3 y}{\partial x^3}(0,t) + \mu\frac{\partial^4 y}{\partial x^3 \partial t}(0,t) + \alpha\frac{\partial^2 y}{\partial t^2}(0,t) + \lambda Q(t) = 0, && 0 \le t \le T, \\
& \frac{\partial y}{\partial x}(0,t) = 0, \quad \frac{\partial^2 y}{\partial x^2}(l,t) = 0, \quad \frac{\partial^3 y}{\partial x^3}(l,t) = 0, && 0 \le t \le T, \\
& y(x,0) = y_0(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = y_1(x), && 0 \le x \le l.
\end{aligned} \tag{4.7}$$

4.3 Adjoint System

To each solution of (4.7), we consider its adjoint system:

$$\begin{aligned}
& m\frac{\partial^2 \theta}{\partial t^2}(x,t) - \mu D\frac{\partial^5 \theta}{\partial x^4 \partial t}(x,t) + D\frac{\partial^4 \theta}{\partial x^4}(x,t) = 0, && 0 \le x \le l,\ 0 \le t \le T, \\
& \frac{\partial^3 \theta}{\partial x^3}(0,t) - \mu\frac{\partial^4 \theta}{\partial x^3 \partial t}(0,t) + \alpha\frac{\partial^2 \theta}{\partial t^2}(0,t) = 0, && 0 \le t \le T, \\
& \frac{\partial \theta}{\partial x}(0,t) = 0, \quad \frac{\partial^2 \theta}{\partial x^2}(l,t) = 0, \quad \frac{\partial^3 \theta}{\partial x^3}(l,t) = 0, && 0 \le t \le T, \\
& \theta(x,0) = \theta_0(x) \quad \text{and} \quad \frac{\partial \theta}{\partial t}(x,0) = \theta_1(x), && 0 \le x \le l,
\end{aligned} \tag{4.8}$$

in which the differential operators are the adjoint of that in the given system on their respective domains. Let $\{\theta_0, \theta_1\} \in F \doteq L^2[0,l] \times H^{-2}[0,l]$ ($H^{-2}[0,l]$ is the dual space of $H^2[0,l]$) with $\theta_0(0) = 0$, $\theta_1(0) = 0$, then we know with small μ that there is a unique solution (cf. Showalter [81]) of (4.8). For $\mu = 0$, the system coincides with the Petrowsky equation with pure elastic vibration having unique solution (cf. Lions and Magenes [53], Showalter [81]) for every $\{\theta_0, \theta_1\} \in F$. Existence of smooth solutions ensures that the solution $\theta(x,t)$ of (4.8) tends to pure elastic solution $\theta_e(x,t)$ when $\mu \to 0+$, and obviously the difference of the solution $|\theta(x,t) - \theta_e(x,t)|$ is bounded above by a positive quantity proportional to μ, which tends to zero as $\mu \to 0+$. Thus the solution $\theta(x,t)$ of (4.8) is a continuous function of μ near zero.

The total energy $E(t)$ at time t of the system (4.8) is defined as

$$E(t) = \frac{1}{2}\int_0^l \Big[m\Big(\frac{\partial\theta}{\partial t}\Big)^2 + D\Big(\frac{\partial^2\theta}{\partial x^2}\Big)^2\Big]dx + \frac{1}{2}m_h\Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2. \tag{4.9}$$

We proceed, with the differentiation of (4. 9) with respect to t and replace $m(\partial^2\theta/\partial t^2)$ by $-D(\partial^4\theta/\partial x^4) + \mu D(\partial^5\theta/\partial x^4\partial t)$, to obtain

$$\begin{aligned}\frac{dE}{dt} &= D\int_0^l \frac{\partial}{\partial x}\Big[\frac{\partial^2\theta}{\partial t\partial x}\frac{\partial^2}{\partial x^2}\Big(\theta - \mu\frac{\partial\theta}{\partial t}\Big) - \frac{\partial\theta}{\partial t}\frac{\partial^3}{\partial x^3}\Big(\theta - \mu\frac{\partial\theta}{\partial t}\Big)\Big]dx \\ &+\mu\int_0^l D\Big(\frac{\partial^3\theta}{\partial x^2\partial t}\Big)^2 dx + m_h\frac{\partial\theta}{\partial t}(0,t)\frac{\partial^2\theta}{\partial t^2}(0,t).\end{aligned}$$

Applying the boundary conditions of (4.8), we get

$$\frac{dE}{dt} = \mu\int_0^l D\Big(\frac{\partial^3\theta}{\partial x^2\partial t}\Big)^2 dx, \tag{4.10}$$

since $m_h = D\alpha$. Integrating (4.10) from 0 to t, we obtain the energy integral as

$$E(t) = E(0) + \mu\int_0^l\int_0^t D\Big(\frac{\partial^3\theta}{\partial x^2\partial\tau}\Big)^2 dx d\tau, \tag{4.11}$$

where

$$E(0) = \frac{1}{2}\int_0^l \Big[m\Big(\frac{\partial\theta}{\partial t}(x,0)\Big)^2 + D\Big(\frac{\partial^2\theta}{\partial x^2}(x,0)\Big)^2\Big]dx. \tag{4.12}$$

Again integrating (4.11) from 0 to T, we obtain the result

$$\begin{aligned}&\frac{1}{2}\int_0^l\int_0^T \Big[m\Big(\frac{\partial\theta}{\partial t}\Big)^2 + D\Big(\frac{\partial^2\theta}{\partial x^2}\Big)^2\Big]dxdt + \frac{1}{2}m_h\int_0^T \Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2 dt \\ &= TE(0) + \mu\int_0^l\int_0^T\int_0^t D\Big(\frac{\partial^3\theta}{\partial x^2\partial\tau}\Big)^2 dx d\tau dt.\end{aligned} \tag{4.13}$$

The integral term in the right hand side of (4.10) shows that some energy is generated throught the system which appears due to the presence of internal distributed force $-\mu D(\partial^5\theta/\partial x^4\partial t)$ in the adjoint system (4.8). Since we assume that that the effect of the term due to internal Voigt-type damping is very small compared to the pure elastic term, the effect of the distributed force $-\mu D(\partial^5\theta/\partial x^4\partial t)$ in (4.8) will also be very small relative to that of pure elastic force $D(\partial^4\theta/\partial x^4)$ in (4.8). Hence the energy due to the internal distributed force is much smaller than that of pure elastic force. Thus we can write from (4.13) that there exists a real $\mu_1 > 0$, such that

$$\frac{1}{2}\int_0^l\int_0^T D\Big(\frac{\partial^2\theta}{\partial x^2}\Big)^2 dxdt \geq \mu\int_0^l\int_0^T\int_0^t D\Big(\frac{\partial^3\theta}{\partial x^2\partial\tau}\Big)^2 dx d\tau dt \quad \text{for } \mu < \mu_1. \tag{4.14}$$

Also when $\mu = 0$, we obtain from (4.11) that $E(t) = \text{constant} = E(0)$ and the adjoint system is energy conserving.

4.4 Backward System and Operator Λ

Let us now consider a time backward system:

$$
\begin{aligned}
& m\frac{\partial^2\phi}{\partial t^2}(x,t)+\mu D\frac{\partial^5\phi}{\partial x^4\partial t}(x,t)+D\frac{\partial^4\phi}{\partial x^4}(x,t)=0, \qquad 0\le x\le l,\ 0\le t\le T,\\
& \frac{\partial^3\phi}{\partial x^3}(0,t)+\mu\frac{\partial^4\phi}{\partial x^3\partial t}(0,t)+\alpha\frac{\partial^2\phi}{\partial t^2}(0,t)+\lambda\beta_0\Big[\theta(0,t)-\mu\frac{\partial\theta}{\partial t}(0,t)\Big]=0,\\
& \frac{\partial\phi}{\partial x}(0,t)=0, \qquad \frac{\partial^2\phi}{\partial x^2}(l,t)=0, \qquad \frac{\partial^3\phi}{\partial x^3}(l,t)=0, \qquad 0\le t\le T,\\
& \phi(x,T)=0 \quad \text{and} \quad \frac{\partial\phi}{\partial t}(x,T)=0, \qquad 0\le x\le l,
\end{aligned}
\tag{4.15}
$$

where, $\beta_0>0$ is a constant, independent of t. It follows from the nonhomogeneous boundary value problem (4.15) that the solution $\phi(x,t)$ depends on $\theta(0,t)$ and $(\partial\theta/\partial t)(0,t)$, and hence on the properties of the initial values $\{\theta_0,\theta_1\}$ of the system (4.8). Now for a given $\{\theta_0,\ \theta_1\}$ in the Hilbert space F, the system (4.15) has a solution $\phi(x,t)$, as in the case of the system (4.8). Therefore the functions $\phi(x,0)$ and $(\partial\phi/\partial t)(x,0)$ can be easily obtained. We then define an operator Λ uniquly by

$$
\Lambda\{\theta_0,\theta_1\}=\{\phi_1,-(\phi_0+\mu\phi_1)\}
\tag{4.16}
$$

where $\phi_0=\phi(x,0)$ and $\phi_1=(\partial\phi/\partial t)(x,0)$.

As in the previous Chapter, we will now estimate the functional

$$
\{\theta_0,\theta_1\}\to\big\langle\Lambda\{\theta_0,\theta_1\},\{\theta_0,\theta_1\}\big\rangle
$$

given by

$$
\big\langle\Lambda\{\theta_0,\theta_1\},\{\theta_0,\theta_1\}\big\rangle=\int_0^l\Big(\theta_0\phi_1-\theta_1(\phi_0+\mu\phi_1)\Big)dx.
\tag{4.17}
$$

To estimate (4.17), we multiply (4.8) by $\phi+\mu(\partial\phi/\partial t)$ and (4.15) by $\theta-\mu(\partial\theta/\partial t)$, integrate over $[0,l]\times[0,T]$ and then subtract :

$$
\begin{aligned}
& m\int_0^l\int_0^T\frac{\partial}{\partial t}\Big(\frac{\partial\theta}{\partial t}\phi-\theta\frac{\partial\phi}{\partial t}\Big)dxdt+m\mu\int_0^l\int_0^T\frac{\partial}{\partial t}\Big(\frac{\partial\theta}{\partial t}\frac{\partial\phi}{\partial t}\Big)dxdt\\
& +D\int_0^l\int_0^T\frac{\partial}{\partial x}\Big[\Big(\phi+\mu\frac{\partial\phi}{\partial t}\Big)\frac{\partial^3}{\partial x^3}\Big(\theta-\mu\frac{\partial\theta}{\partial t}\Big)-\Big(\theta-\mu\frac{\partial\theta}{\partial t}\Big)\frac{\partial^3}{\partial x^3}\Big(\phi+\mu\frac{\partial\phi}{\partial t}\Big)\Big]dxdt\\
& -D\int_0^l\int_0^T\frac{\partial}{\partial x}\Big[\frac{\partial^2}{\partial x^2}\Big(\theta-\mu\frac{\partial\theta}{\partial t}\Big)\frac{\partial}{\partial x}\Big(\phi+\mu\frac{\partial\phi}{\partial t}\Big)-\frac{\partial}{\partial x}\Big(\theta-\mu\frac{\partial\theta}{\partial t}\Big)\frac{\partial^2}{\partial x^2}\Big(\phi+\mu\frac{\partial\phi}{\partial t}\Big)\Big]dxdt=0.
\end{aligned}
$$

Utilization of the initial and the boundary conditions of the systems (4.8) and (4.15), and then after a straightforward calculation, the above yields

$$\int_0^l (\theta_0\phi_1 - \theta_1\phi_0 - \mu\theta_1\phi_1)dx = C\int_0^T \Big[\theta(0,t) - \mu\frac{\partial\theta}{\partial t}(0,t)\Big]^2 dt, \tag{4.18}$$

where $C = \beta_0/m$. Thus the functional (4.17) is obtained as

$$\Big\langle \Lambda\{\theta_0,\theta_1\},\{\theta_0,\theta_1\}\Big\rangle = C\int_0^T \Big[\theta(0,t) - \mu\frac{\partial\theta}{\partial t}(0,t)\Big]^2 dt. \tag{4.19}$$

For $T > T_0$ and $\mu < \mu_0$, we shall show that the right hand side of (4.19) defines a norm on the initial values $\{\theta_0,\theta_1\}$, equivalent to the norm of the spcae F, i.e.,

$$\Big\|\{\theta_0,\theta_1\}\Big\|_F^2 = C\int_0^T \Big[\theta(0,t) - \mu\frac{\partial\theta}{\partial t}(0,t)\Big]^2 dt. \tag{4.20}$$

4.5 Estimate of the Least Control Time T_0

We first establish the following results.

Lemma 4.1. Let $\theta(x,t)$ be a solution of the system (4.8), then

$$\begin{aligned}\int_0^T X(t)dt &= TX(0) + \mu\frac{m}{m_h}\int_0^T\int_0^t D\Big(\frac{\partial^4\theta}{\partial x^3\partial\tau}(0,\tau)\Big)^2 d\tau dt\\ &\quad + \mu\int_0^l\int_0^T\int_0^t D\Big(\frac{\partial^5\theta}{\partial x^4\partial\tau}\Big)^2 dx d\tau dt,\end{aligned} \tag{4.21}$$

where

$$X(t) = \frac{1}{2}\int_0^l\Big[m\Big(\frac{\partial^3\theta}{\partial x^2\partial t}\Big)^2 + D\Big(\frac{\partial^4\theta}{\partial x^4}\Big)^2\Big]dx + \frac{1}{2}\frac{m}{m_h}D\Big(\frac{\partial^3\theta}{\partial x^3}(0,t)\Big)^2. \tag{4.22}$$

Proof. Differentiating (4.22) with respect to t and then integrating by parts, we obtain

$$\begin{aligned}\frac{dX}{dt} &= m\Big[\frac{\partial^3\theta}{\partial x^2\partial t}\frac{\partial^3\theta}{\partial x\partial t^2} - \frac{\partial^4\theta}{\partial x^3\partial t}\frac{\partial^2\theta}{\partial t^2}\Big]_0^l\\ &\quad + \int_0^l\Big(m\frac{\partial^2\theta}{\partial t^2} + D\frac{\partial^4\theta}{\partial x^4}\Big)\frac{\partial^5\theta}{\partial x^4\partial t}dx + \frac{m}{m_h}D\frac{\partial^3\theta}{\partial x^3}(0,t)\frac{\partial^4\theta}{\partial x^3\partial t}(0,t).\end{aligned}$$

On application of the system (4.8), it becomes

$$\frac{dX}{dt} = \mu\frac{m}{m_h}D\Big(\frac{\partial^4\theta}{\partial x^3\partial t}(0,t)\Big)^2 + \mu\int_0^l D\Big(\frac{\partial^5\theta}{\partial x^4\partial t}\Big)^2 dx, \tag{4.23}$$

since $\alpha = m_h/D$. Integrating (4.23) from 0 to t, we obtain

$$X(t) = X(0) + \mu\frac{m}{m_h}\int_0^t D\Big(\frac{\partial^4\theta}{\partial x^3\partial\tau}(0,\tau)\Big)^2 d\tau + \mu\int_0^l\int_0^t D\Big(\frac{\partial^5\theta}{\partial x^4\partial\tau}\Big)^2 dx d\tau. \tag{4.24}$$

Again on integration of (4.24) from 0 to T, the Lemma follows immediately.

It is clear that the terms in the right hand side of (4.23) arise due to effect of μ dependent term in the system (4.8), so the last two terms in the right hand side of (4.21) will be much smaller than that of the corresponding pure elastic terms. Therefore, from the statements following equations (4.1) and (4.3) we can write for (4.21), that there exists small real quantities $\mu_2,\ \mu_3 > 0$ such that

$$\mu\int_0^l\int_0^T\int_0^t D\Big(\frac{\partial^5\theta}{\partial x^4\partial\tau}\Big)^2 dx d\tau dt \le \frac{1}{2}\int_0^l\int_0^T D\Big(\frac{\partial^4\theta}{\partial x^4}\Big)^2 dx dt \quad \text{for } \mu < \mu_2, \tag{4.25}$$

and

$$\mu\int_0^T\int_0^t D\Big(\frac{\partial^4\theta}{\partial x^3\partial\tau}(0,\tau)\Big)^2 d\tau dt \le \frac{1}{3}\int_0^T D\Big(\frac{\partial^3\theta}{\partial x^3}(0,t)\Big)^2 dt \quad \text{for } \mu < \mu_3. \tag{4.26}$$

Lemma 4.2. Let

$$Y(t) = m\int_0^l (l-x)\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial t}dx \text{ and } Z(t) = D\int_0^l (l-x)\frac{\partial^2\theta}{\partial x\partial t}\frac{\partial^4\theta}{\partial x^4}dx, \tag{4.27}$$

then

$$|Y(t)| \le \frac{2l^2}{\pi}\sqrt{\frac{m}{D}}E(t) \tag{4.28}$$

and

$$|Z(t)| \le \frac{l^2}{\pi}\sqrt{\frac{D}{m}}\Big[2X(t) - \frac{m}{m_h}D\Big(\frac{\partial^3\theta}{\partial x^3}(0,t)\Big)^2\Big]. \tag{4.29}$$

Proof. Since $(\partial\theta/\partial x)(0,t) = 0$, from Wirtinger's inequality (cf. [80]), we can write

$$\int_0^l \Big(\frac{\partial\theta}{\partial x}\Big)^2 dx \le \frac{4l^2}{\pi^2}\int_0^l \Big(\frac{\partial^2\theta}{\partial x^2}\Big)^2 dx, \tag{4.30}$$

and

$$\int_0^l \Big(\frac{\partial^2\theta}{\partial x\partial t}\Big)^2 dx \le \frac{4l^2}{\pi^2}\int_0^l \Big(\frac{\partial^3\theta}{\partial x^2\partial t}\Big)^2 dx. \tag{4.31}$$

Using the inequality

$$|ab| \le \frac{1}{2}\Big(\epsilon a^2 + \frac{1}{\epsilon}b^2\Big) \quad \text{for any real positive } \epsilon, \tag{4.32}$$

we have from (4.27)

$$|Y(t)| \leq \frac{l^2}{\pi}\sqrt{\frac{m}{D}}\int_0^l \Big[m\Big(\frac{\partial\theta}{\partial t}\Big)^2 + D\frac{\pi^2}{4l^2}\Big(\frac{\partial\theta}{\partial x}\Big)^2\Big]dx \tag{4.33}$$

and

$$|Z(t)| \leq \frac{l^2}{\pi}\sqrt{\frac{D}{m}}\int_0^l \Big[m\frac{\pi^2}{4l^2}\Big(\frac{\partial^2\theta}{\partial x\partial t}\Big)^2 + D\Big(\frac{\partial^4\theta}{\partial x^4}\Big)^2\Big]dx. \tag{4.34}$$

By the inequalities (4.30) and (4.31), the Lemma follows from the above relations (4.33) and (4.34).

Now we shall need to establish the following two inequalities alongwith the estimation of least control time T_0, which finally helps to prove the the exact controllability result of the system (4.7) by HUM.

There exist positive numbers A, B independent of μ and K, T_0, such that

$$TE(0) \geq \frac{m_h}{2}\int_0^T \Big[\frac{\partial\theta}{\partial t}(0,t) - \mu\frac{\partial^2\theta}{\partial t^2}(0,t)\Big]^2 dt \geq \frac{1 - A\mu - B\mu^2}{1+K}(T - T_0)E(0). \tag{4.35}$$

By multiplier technique to establish (4.35), we multiply the first equation of (4.8) by $(l-x)(\partial\theta/\partial x)$ and integrate by parts over $[0,l]\times[0,T]$, and then obtain

$$m\Big[\int_0^l (l-x)\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial t}dx\Big]_0^T - \frac{1}{2}\int_0^l\int_0^T (l-x)\frac{\partial}{\partial x}\Big[m\Big(\frac{\partial\theta}{\partial t}\Big)^2 + D\Big(\frac{\partial^2\theta}{\partial x^2}\Big)^2\Big]dxdt$$
$$+D\int_0^l\int_0^T \frac{\partial\theta}{\partial x}\frac{\partial^3\theta}{\partial x^3}dxdt = \mu D\Big[\int_0^l (l-x)\frac{\partial\theta}{\partial x}\frac{\partial^4\theta}{\partial x^4}dx\Big]_0^T - \mu D\int_0^l\int_0^T (l-x)\frac{\partial^2\theta}{\partial x\partial t}\frac{\partial^4\theta}{\partial x^4}dxdt,$$

where we have used the boundary conditions in (4.8). Again integrating by parts, the above can be written as

$$\frac{l}{2}\int_0^T \Big[m\Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2 + D\Big(\frac{\partial^2\theta}{\partial x^2}(0,t)\Big)^2\Big]dt = \frac{1}{2}\int_0^l\int_0^T \Big[m\Big(\frac{\partial\theta}{\partial t}\Big)^2 + D\Big(\frac{\partial^2\theta}{\partial x^2}\Big)^2\Big]dxdt$$
$$-\big[Y(t)\big]_0^T - \mu\int_0^T Z(t)dt - D\int_0^T\Big[\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}\Big]_0^l dt + D\int_0^l\int_0^T\Big(\frac{\partial^2\theta}{\partial x^2}\Big)^2 dxdt$$
$$+\mu D\Big[\Big[(l-x)\frac{\partial\theta}{\partial x}\frac{\partial^3\theta}{\partial x^3}\Big]_0^l - \int_0^l\Big((l-x)\frac{\partial^2\theta}{\partial x^2} - \frac{\partial\theta}{\partial x}\Big)\frac{\partial^3\theta}{\partial x^3}dx\Big]_0^T.$$

By the use of the boundary conditions in (4.8), the energy integral (4.13) and the Lemma 4.2, the above becomes

$$\frac{l}{2}\int_0^T \Big[m\Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2 + D\Big(\frac{\partial^2\theta}{\partial x^2}(0,t)\Big)^2\Big]dt + \frac{1}{2}m_h\int_0^T\Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2 dt \geq TE(0) - \big[Y(t)\big]_0^T$$
$$-\mu\frac{l^2}{\pi}\sqrt{\frac{D}{m}}\int_0^T\Big[2X(t) - \frac{m}{m_h}D\Big(\frac{\partial^3\theta}{\partial x^3}(0,t)\Big)^2\Big]dt - \mu\Big[\int_0^l D\Big(\frac{\partial^2\theta}{\partial x^2}\Big)^2 dx\Big]_0^T. \tag{4.36}$$

Now from (4.28) we can write

$$\begin{aligned} \Big|Y(t)\Big|_0^T &\le \frac{2l^2}{\pi}\sqrt{\frac{m}{D}}\big[E(T)+E(0)\big] \\ &= 4\frac{l^2}{\pi}\sqrt{\frac{m}{D}}E(0)+\mu\frac{2l^2}{\pi}\sqrt{\frac{m}{D}}\int_0^l\int_0^T D\Big(\frac{\partial^3\theta}{\partial x^2\partial t}\Big)^2 dxdt \end{aligned} \tag{4.37}$$

by (4.11). Again we have

$$\begin{aligned} -2E(0) &\le \Big[\int_0^l D\Big(\frac{\partial^2\theta}{\partial x^2}\Big)^2 dx\Big]_0^T \le 2E(T)-m_h\Big[\frac{\partial\theta}{\partial t}(0,T)\Big]^2 \\ &= 2E(0)+2\mu\int_0^l\int_0^T D\Big(\frac{\partial^3\theta}{\partial x^2\partial t}\Big)^2 dxdt - m_h\Big[\frac{\partial\theta}{\partial t}(0,T)\Big]^2. \end{aligned} \tag{4.38}$$

Introducing (4.37) and (4.38) into (4.36), yields

$$\begin{aligned} &\frac{l}{2}\int_0^T\Big[m\Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2+D\Big(\frac{\partial^2\theta}{\partial x^2}(0,t)\Big)^2\Big]dt+\frac{1}{2}m_h\int_0^T\Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2dt \ge \\ &\quad\Big[T-\Big(\frac{4l^2}{\pi}\sqrt{\frac{m}{D}}+2\mu\Big)\Big]E(0)-\mu\Big(\frac{3l^2}{\pi}\sqrt{\frac{D}{m}}+2\mu\frac{D}{m}\Big)\int_0^T\Big[2X(t)-\frac{m}{m_h}D\Big(\frac{\partial^3\theta}{\partial x^3}(0,t)\Big)^2\Big]dt \\ &\quad+2\mu\Big(\frac{l^2}{\pi}\sqrt{\frac{D}{m}}+\mu\frac{D}{m}\Big)\int_0^l\int_0^T D\Big(\frac{\partial^4\theta}{\partial x^4}\Big)^2dxdt+\mu m_h\Big[\frac{\partial\theta}{\partial t}(0,T)\Big]^2 \end{aligned}$$

with the help of (4.22). Applying Lemma 4.1, it leads to

$$\begin{aligned} &\frac{l}{2}\int_0^T\Big[m\Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2+D\Big(\frac{\partial^2\theta}{\partial x^2}(0,t)\Big)^2\Big]dt+\frac{1}{2}m_h\int_0^T\Big(\frac{\partial\theta}{\partial t}(0,t)\Big)^2dt-\mu m_h\Big[\frac{\partial\theta}{\partial t}(0,T)\Big]^2 \\ &\ge\Big[T-\Big(\frac{4l^2}{\pi}\sqrt{\frac{m}{D}}+2\mu\Big)\Big]E(0)-2\mu\Big(\frac{3l^2}{\pi}\sqrt{\frac{D}{m}}+2\mu\frac{D}{m}\Big)TX(0) \\ &\quad+\mu\frac{m}{m_h}\Big(\frac{3l^2}{\pi}\sqrt{\frac{D}{m}}+2\mu\frac{D}{m}\Big)\Big[\int_0^T D\Big(\frac{\partial^3\theta}{\partial x^3}(0,t)\Big)^2dt-2\mu\int_0^T\int_0^t D\Big(\frac{\partial^4\theta}{\partial x^3\partial\tau}(0,\tau)\Big)^2d\tau dt\Big] \\ &\quad+2\mu\Big(\frac{l^2}{\pi}\sqrt{\frac{D}{m}}+\mu\frac{D}{m}\Big)\Big[\int_0^l\int_0^T D\Big(\frac{\partial^4\theta}{\partial x^4}\Big)^2dxdt-3\mu\int_0^l\int_0^T\int_0^t D\Big(\frac{\partial^5\theta}{\partial x^4\partial\tau}\Big)^2dxd\tau dt\Big] \end{aligned} \tag{4.39}$$

Let us now we define a constant $K_0>0$ independent of μ, such that

$$X(0)\le K_0E(0). \tag{4.40}$$

Hence from (4.39) we can write by virtue of (4.25) and (4.26)

$$\frac{1}{2}m_h\int_0^T\Big[\frac{\partial\theta}{\partial t}(0,t)-\mu\frac{\partial^2\theta}{\partial t^2}(0,t)\Big]^2dt\ge\frac{1}{1+K}(1-A\mu-B\mu^2)(T-T_0)E(0), \tag{4.41}$$

where

$$K = \frac{l\int_0^T \left[m\left(\frac{\partial\theta}{\partial t}(0,t)\right)^2 + D\left(\frac{\partial^2\theta}{\partial x^2}(0,t)\right)^2\right]dt}{m_h\int_0^T \left[\frac{\partial\theta}{\partial t}(0,t) - \mu\frac{\partial^2\theta}{\partial t^2}(0,t)\right]^2 dt} > 0, \tag{4.42}$$

$$A = \frac{6l^2K_0}{\pi}\sqrt{\frac{D}{m}}, \qquad B = 4K_0\frac{D}{m} \tag{4.43}$$

and

$$T_0 = \frac{\frac{4l^2}{\pi}\sqrt{\frac{m}{D}} + 2\mu}{1 - A\mu - B\mu^2}. \tag{4.44}$$

To obtain the reverse inequality in (4.35), we have from (4.13)

$$\begin{aligned} \frac{1}{2}m_h\int_0^T \left[\frac{\partial\theta}{\partial t}(0,t) - \mu\frac{\partial^2\theta}{\partial t^2}(0,t)\right]^2 dt &\le TE(0) \\ &-\frac{1}{2}\left[\int_0^l\int_0^T m\left(\frac{\partial\theta}{\partial t}\right)^2 dxdt - \mu^2\alpha\int_0^T D\left(\frac{\partial^2\theta}{\partial t^2}(0,t)\right)^2 dt\right] \\ &-\frac{1}{2}\left[\int_0^l\int_0^T D\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dxdt - 2\mu\int_0^l\int_0^T\int_0^t D\left(\frac{\partial^3\theta}{\partial x^2\partial\tau}\right)^2 dxd\tau dt\right]. \end{aligned} \tag{4.45}$$

We note that the integrals in the expression

$$\int_0^l\int_0^T m\left(\frac{\partial\theta}{\partial t}\right)^2 dxdt - \mu^2\alpha\int_0^T D\left(\frac{\partial^2\theta}{\partial t^2}(0,t)\right)^2 dt \tag{4.46}$$

tend to the corresponding integrals for pure elastic case $\theta_e(x,t)$ as $\mu \to 0+$. Hence by choosing $\mu < \mu_4$ where $\mu_4 > 0$, the expression (4.46) can be made nonnegative. Therefore, using the relation (4.14) the left inequality in (4.35) follows immediately from (4.45).

It is obvious that the roots of the quadratic equation $1 - A\mu - B\mu^2 = 0$ are always real, and one is positive, say μ_5, and other negative. We observe that between these two roots the expression $1 - A\mu - B\mu^2$ is always positive. Since $\mu > 0$ is assumed to be small enough, the analysis of the above holds for $\mu < \mu_0$ where μ_0 is given by $\mu_0 = \min_{1\le i\le 5}\{\mu_i\}$.

We now define a premitive function $\psi(x,t)$ satisfying the indefinite integral

$$\psi(x,t) = \int^t \theta(x,t)dt \tag{4.47}$$

Then we have

$$\begin{aligned}m\frac{\partial^2\psi}{\partial t^2}(x,t) &- \mu D\frac{\partial^5\psi}{\partial x^4\partial t}(x,t) + D\frac{\partial^4\psi}{\partial x^4}(x,t)\\ &= m\frac{\partial\theta}{\partial t}(x,t) - \mu D\frac{\partial^4\theta}{\partial x^4}(x,t) + D\int^t \frac{\partial^4\theta}{\partial x^4}(x,t)dt\\ &= m\frac{\partial\theta}{\partial t}(x,t) - \mu D\frac{\partial^4\theta}{\partial x^4}(x,t) - \int^t \Big[m\frac{\partial^2\theta}{\partial t^2} - \mu D\frac{\partial^5\theta}{\partial x^4\partial t}\Big]dt\\ &= 0.\end{aligned}$$

Thus $\psi(x,t)$ satisfies the following system:

$$\begin{aligned}&m\frac{\partial^2\psi}{\partial t^2}(x,t) - \mu D\frac{\partial^5\psi}{\partial x^4\partial t}(x,t) + D\frac{\partial^4\psi}{\partial x^4}(x,t) = 0, && 0\le x\le l,\ 0\le t\le T,\\ &\frac{\partial^3\psi}{\partial x^3}(0,t) - \mu\frac{\partial^4\psi}{\partial x^3\partial t}(0,t) + \alpha\frac{\partial^2\psi}{\partial t^2}(0,t) = 0, && 0\le t\le T,\\ &\frac{\partial\psi}{\partial x}(0,t) = 0,\quad \frac{\partial^2\psi}{\partial x^2}(l,t) = 0\quad \frac{\partial^3\psi}{\partial x^3}(l,t) = 0, && 0\le t\le T,\\ &\psi(x,0) = \int^t \theta(x,t)dt\Big|_{t=0} = \psi_0\ \text{(say)}, && 0\le x\le l,\\ &\frac{\partial\psi}{\partial t}(x,0) = \theta(x,0) = \theta_0 = \psi_1\ \text{(say)}, && 0\le x\le l,\end{aligned} \tag{4.48}$$

Thus the system (4.48) is analogous to the system (4.8) and so the inequalities in (4.35) can be applicable for $\psi(x,t)$. Since $(\partial\psi/\partial t)(x,t) = \theta(x,t)$, using the inequalities in (4.35) for the solution of the system (4.48), we have

$$\begin{aligned}\frac{T}{2}\int_0^l \Big[m\Big(\frac{\partial\psi}{\partial t}(x,0)\Big)^2 + D\Big(\frac{\partial^2\psi}{\partial x^2}(x,0)\Big)^2\Big]dx &\ge \frac{m_h}{2}\int_0^T \Big[\frac{\partial\psi}{\partial t}(0,t) - \mu\frac{\partial^2\psi}{\partial t^2}(0,t)\Big]^2 dt\\ &\ge \frac{1 - A\mu - B\mu^2}{1+K}(T - T_0)\frac{1}{2}\int_0^l \Big[m\Big(\frac{\partial\psi}{\partial t}(x,0)\Big)^2 + D\Big(\frac{\partial^2\psi}{\partial x^2}(x,0)\Big)^2\Big]dx.\end{aligned} \tag{4.49}$$

By Poincare inequality (cf. Aubin [1]), we have that the norm

$$\Big|\Big|\frac{\partial^2\psi}{\partial x^2}\Big|\Big|^2_{L^2[0,l]} = \int_0^l \Big(\frac{\partial^2\psi}{\partial x^2}\Big)^2 dx$$

is equivalent to the norm of ψ in $H^2[0,l]$. Therefore, there exists positive numbers C_0 and C_1 such that (4.49) can be written as

$$\begin{aligned}C_1T\Big[||\psi_1||^2_{L^2[0,l]} + ||\psi_0||^2_{H^2[0,l]}\Big] &\ge C\int_0^T \Big[\frac{\partial\psi}{\partial t}(0,t) - \mu\frac{\partial^2\psi}{\partial t^2}(0,t)\Big]^2 dt\\ &\ge C_0(1 - A\mu - B\mu^2)(T - T_0)\Big[||\psi_1||^2_{L^2[0,l]} + ||\psi_0||^2_{H^2[0,l]}\Big].\end{aligned} \tag{4.50}$$

Hence we have from (4.50),

$$C_1 T\Big[||\theta_0||^2_{L^2[0,l]} + ||\theta_1||^2_{H^{-2}[0,l]}\Big] \geq C\int_0^T \Big[\theta(0,t) - \mu\frac{\partial\theta}{\partial t}(0,t)\Big]^2 dt$$
$$\geq C_0(1 - A\mu - B\mu^2)(T - T_0)\Big[||\theta_0||^2_{L^2[0,l]} + ||\theta_1||^2_{H^{-2}[0,l]}\Big], \qquad (4.51)$$

in virtue of (4.47).

4.6 Exact Controllability Result

In the literature, exact controllability of a system means: for a given time $T > 0$, to find a suitable control function which drives the whole system to a desired final state or rest at the time T. To study the exact controllability of the system (4.7) at some finite time $T > 0$, we require to select a control force $Q(t)$ appropriately on $[0, T]$ such that the system (4.7), would be driven to rest (the desired final state) at time $t = T$. Then the solution of the system (4.7) must satisfy

$$y(x,T) = 0 \quad \text{and} \quad \frac{\partial y}{\partial t}(x,T) = 0. \qquad (4.52)$$

The result (4.52) for exact controllability of the system (4.7) follows from the Theorem:

Theorem 4.1. Let $T > T_0$ and $\mu < \mu_0$, then for every $y_0 \in H^2[0,l]$ and $y_1 \in L^2[0,l]$, there is a control function $Q(t) \in L^2[0,T]$ such that $y(x,t)$, the solution of the system (4.7) satisfies (4.52).

Proof. The inequalities in (4.51), implies that for $T > T_0$ and $\mu < \mu_0$, (4.19) defines a norm of $\{\theta_0, \theta_1\}$ which is equivalent to the norm on the Hilbert space F. We can conclude from the Lax-Milgram theorem (cf. Aubin [1]) in virtue of (4.17) and (4.19) that Λ is an isomorphism from F to F'. Hence for given $\{y_0, y_1\} \in F'$, there exists $\{\theta_0, \theta_1\} \in F$ such that

$$\Lambda\{\theta_0, \theta_1\} = \{y_1, -(y_0 + \mu y_1)\}. \qquad (4.53)$$

Now if we take the control $Q(t)$ in (4.3) of the original problem as proportional to $\big[\theta(0,t) - \mu(\partial\theta/\partial t)(0,t)\big]$ say, $Q(t) = \beta_0\big[\theta(0,t) - \mu(\partial\theta/\partial t)(0,t)\big]$, where $\theta(x,t)$ is the solution of (4.8) with $\{\theta_0, \theta_1\}$ as solution of (4.53), then from (4.16), we have that the function $\phi(x,t)$, the solution of the system (4.15), satisfies $\phi_0 = y_0$ and $\phi_1 = y_1$. By the uniqueness theorem we finally conclude that

$$\phi(x,t) = y(x,t) \qquad 0 \leq x \leq l,\ 0 \leq t \leq T, \qquad (4.54)$$

and the result (4.52) for exact controllability of the original system (4.7) then ensures from the system (4.15).

4.7 Concluding Remarks

From the obsevability of the adjoint system, it follows that the original system (4.7) with small μ $(0 < \mu < \mu_0)$ is exactly controllable for $T > T_0$ (cf. [19]). Thus we can reveal the time T_0 as the estimated least time for exact control of the vibrations of the hybrid elastic panel. Equation (4.44) shows explicit dependence of T_0 on μ proving that small viscous damping has the effect of increasing this time over the purely elastic case. This can be rendered by the resistance on the propagation of control wave through the viscoelastic flexible panel. The method as well as the result are analogously valid for Euler-Bernoulli beam also; in that case the flexural rigidity EI of the beam takes the place of D of the panel.

CHAPTER 5

SPACE-TIME GALERKIN APPROXIMATION FOR EXACT CONTROL OF FLEXURAL VIBRATIONS*

5.1 Introduction

The theoretical approach on the distributed parameter control problems for exact controllability of torsional vibrations as well as transverse vibrations of the flexible hybrid structure have been developed in the preceding Chapters. Distributed parameter control problems are highly involved, depending on the form of partial differential equations describing the states of the system. The applicable controls in general, may be distributed with time dependent or space variable dependent or may be dependent on both time and space variables and in addition, it may occur only on the boundary. One computational technnique for one type of distributed parameter control problem may work well, but in general the same technique may not work for another type. Furthermore, if a numerical solution is sought, the success of a particular computational technique depends very much on the particular computational algorithm used, for the solution of the partial differential equations. Because of these complications, it is to the fact that there can be hardly a truely unified computational approach to distributed parameter control problems. The computation of the numercal solution of the partial differential equations is thus very demanding. In view of this, the purpose of this Chapter, is to give a comprehensive numerical technique to support the theory treated in the earlier Chapters.

Among the various methods for solving the boundary value problem, a possible approach is to use finite defference approximation to the partial derivatives, although the accuracy of the method is limited. Another approach from engineering point of view is associated with the eigen value problem in which first few modes of the shape functions are taken into consideration. But actually the number of modes are infinite and the number of modes that should be retained is not known *a priori* ; afterall the

*The contents of this chapter have been published in the paper *Exact Controllability of a Linear Euler-Bernoulli Panel* —Gorain and Bose, ' Journal of Sound and Vibration' Vol. 217, 637–652, (1998).

method is not less difficult. In contrast, a semi-analytic Galerkin's residual method (cf. Kantorovich and Krylov [34]) can be applied with respect to both space and time variables to the distributed parameter control system for approximate numerical solution with small computational effort. For solving boundary value problems, the Galerkin's method makes possible a simpler and direct set-up, and at the same time having wider application and dissemination than Ritz's method (cf. [34]), though both methods lead to one and the same approximate solution. However, Ritz's method is not applicable in the presesnce of internal dissipation of the system.

5.2 Space-Time Galerkin Approximate Scheme

Herein we would like to build up a framework on the basis of Galerkin's weighted residual technique [34] for the vibrating control problem of elastic system, particularly the control problem of flexural vibrations of hybrid elastic panel described in Chapter 3, as there is a greater importance of transversely vibrating structures. Application of Galerkin's residual method with respect to space coordinate on initial-boundary value problem (3.8) in Chapter 3, leads to a set of ordinary differential equations and futher application of it with respect to time coordinate finally yields the set of matrix equations giving a closed form approximate numerical scheme of the vibrating control system (3.8). We proceed by constructing admissible approximate displacement function as well as approximate boundary control force that satisfy the final conditions (3.35) in Chapter 3, as closely as possible. For this it is convenient to treat the above boundary value problem in the above two steps.

In the first step, the approximate displacement for the system (3.8) is written as superposition of polynomial shape functions of the following type:

$$y(x,t) = \sum_{i=1}^{p+1} f_i(x)\phi_i(t), \qquad 0 \le x \le l,\ 0 \le t \le T, \tag{5.1}$$

where each

$$f_i = \sum_{j=1}^{n+1} a_j^i \left(\frac{x}{l}\right)^{j-1} \quad (i = 1, 2, ..., p+1) \tag{5.2}$$

is a polynomial in x/l of degree n. The functions f_i satisfy the homogeneous boundary conditions to (3.5)–(3.6) for $i = 1, 2, ..., p$, while f_{p+1} satisfies the non-homogeneous boundary condition (3.4) corresponding to the system (3.8) in Chapter 3. The coefficient functions $\phi_i(t)$ for $i = 1, 2, ..., p$ are to be determined for finding the approximate solution of the system (3.8) by Galerkin technique, while $\phi_{p+1}(t)$ on account of (3.4) is given by

$$\phi_{p+1}(t) = -l^3\Big[\alpha\frac{\partial^2 y}{\partial t^2}(0,t) + \lambda Q(t)\Big], \tag{5.3}$$

where l^3 is a dimesionality constant.

With the above remarks, substituting (5.1) into the governing flexural vibration equation in (3.8), the integral of the weighted residue (with weight f_i) over $[0,l]$ set equal to zero:

$$\int_0^l f_i(x)\Big(m\frac{\partial^2}{\partial t^2} + D\frac{\partial^4}{\partial x^4}\Big)\sum_{j=1}^{p+1} f_j(x)\phi_j(t)dx = 0, \quad (i = 1,2,...,p), \tag{5.4}$$

yields the system of ordinary differential equation

$$\mathbf{A}\ddot{\Phi} + \mathbf{B}\Phi + \mathbf{E}\ddddot{\Phi} = \lambda[\mathbf{C}\ddot{Q}(t) + \mathbf{D}Q(t)]. \tag{5.5}$$

Applying the same weighted residual technique, the initial conditions (3.7) similarly lead to

$$\Phi(0) = \Phi^0 \qquad \text{and} \qquad \dot{\Phi}(0) = \Phi^1. \tag{5.6}$$

where dot represnts time derivative. Introducing the transformation

$$\Phi(t) = \Psi(t) + \Phi^0 + t\Phi^1, \tag{5.7}$$

the above relation (5.5) changes to

$$\mathbf{A}\ddot{\Psi} + \mathbf{B}\Psi + \mathbf{B}(\Phi^0 + t\Phi^1) + \mathbf{E}\ddddot{\Psi} = \lambda[\mathbf{C}\ddot{Q}(t) + \mathbf{D}Q(t)] \tag{5.8}$$

together with (5.6) reduces to homogeneous initial conditions

$$\Psi(0) = 0 \qquad \text{and} \qquad \dot{\Psi}(0) = 0. \tag{5.9}$$

The entries of the square matrices $\mathbf{A}, \mathbf{B}, \mathbf{E}$ are produced as

$$\begin{aligned}
\mathbf{A} &= \Big[ml\sum_{k=1}^{n+1}\sum_{s=1}^{n+1}\frac{a_k^i a_s^j}{s+k-1} - \alpha a_1^j D\sum_{k=1}^{n+1}\sum_{s=5}^{n+1} a_k^i a_s^{p+1}\frac{(s-1)(s-2)(s-3)(s-4)}{s+k-5}\Big]_{p\times p}, \\
\mathbf{B} &= \Big[\frac{D}{l^3}\sum_{k=1}^{n+1}\sum_{s=5}^{n+1} a_k^i a_s^j\frac{(s-1)(s-2)(s-3)(s-4)}{s+k-5}\Big]_{p\times p}, \\
\mathbf{E} &= \Big[-ml^4\alpha a_1^j\sum_{k=1}^{n+1}\sum_{s=1}^{n+1}\frac{a_k^i a_s^{p+1}}{s+k-1}\Big]_{p\times p}
\end{aligned} \tag{5.10}$$

and that of the column vectors $\mathbf{C}, \mathbf{D}, \Phi, \Psi, \Phi^0, \Phi^1$ are as

$$\mathbf{C} = \Big[ml^4 \sum_{k=1}^{n+1} \sum_{s=1}^{n+1} \frac{a_k^i a_s^{p+1}}{s+k-1}\Big]_{p\times 1},$$

$$\mathbf{D} = \Big[D \sum_{k=1}^{n+1} \sum_{s=5}^{n+1} a_k^i a_s^{p+1} \frac{(s-1)(s-2)(s-3)(s-4)}{s+k-5}\Big]_{p\times 1}, \tag{5.11}$$

$$\Phi = \Phi(t) = [\phi_i(t)]_{p\times 1}, \qquad \Psi = \Psi(t) = [\psi_i(t)]_{p\times 1},$$

$$\Phi^0 = [\phi_i(0)]_{p\times 1}, \qquad \Phi^1 = [\dot{\phi}_i(0)]_{p\times 1}$$

In the next step, we repeat the Galerkin's weighted residual method in the time domain for the system (5.8) with homogeneous initial conditions (5.9). As a tool, the approximation of $\Psi(t)$ is written as

$$\Psi(t) = \sum_{k=1}^{q} \mathbf{X}_k \Big(\frac{t}{T}\Big)^{k+1} \tag{5.12}$$

satisfying the boundary conditions (5.9), where each $\mathbf{X}_k$ $(k = 1, 2, ..., q)$ is a $p \times 1$ column vector. The determination of each $\mathbf{X}_k$ $(k = 1, 2, ..., q)$ yields the approximate solution $\Psi(t)$ of (5.12) and hence that of $\Phi(t)$ from (5.7).

Now as in the previous step, we proceed by substituting (5.12) into (5.8) and taking the the integral of the weighted residue with weight $(t/T)^{(r+1)}$ over $[0, T]$ to obtain the matrix equation

$$\sum_{k=1}^{q} \mathbf{M}_{rk} \mathbf{X}_k = -\xi_r(\mathbf{B}\Phi^0) - \eta_r(\mathbf{B}\Phi^1) + \mathbf{Q}_r^*, \qquad (r = 1, 2, ..., q) \tag{5.13}$$

where each $\mathbf{M}_{rk}$ $(r = 1, 2, ..., q;\ k = 1, 2, ..., q)$ is a $p \times p$ square matrix and each $\mathbf{Q}_r^*$ $(r = 1, 2, ..., q)$ is a $p \times 1$ column vector obtained explicitly as

$$\mathbf{M}_{rk} = \Big[\frac{(k+1)k}{T(k+r+1)} \mathbf{A} + \frac{T}{k+r+3} \mathbf{B} + \frac{(k+1)k(k-1)(k-2)}{T^3(k+r-1)} \mathbf{E}\Big], \tag{5.14}$$

$$\mathbf{Q}_r^* = \lambda \int_0^T \big[\mathbf{C}\ddot{Q}(t) + \mathbf{D}Q(t)\big] \Big(\frac{t}{T}\Big)^{r+1} dt \tag{5.15}$$

and

$$\xi_r = \frac{T}{r+2}, \qquad \eta_r = \frac{T^2}{r+3}. \tag{5.16}$$

To solve $\mathbf{X}_k$ from (5.13), we have to invert the matrix $\mathbf{M} = [\mathbf{M}_{rk}]$. Let us suppose that $\mathbf{M}$ is non-singular and $\mathbf{F} = [\mathbf{F}_{kr}]$ the inverse of $\mathbf{M}$, where each $\mathbf{F}_{kr}$ $(k = 1, 2, ..., q;$ $r = 1, 2, ..., q)$ is a $p \times p$ square matrix, then from (5.13) we have the scheme

$$\mathbf{X}_k = -\sum_{r=1}^{q}\xi_r \mathbf{F}_{kr}\mathbf{B}\Phi^0 - \sum_{r=1}^{q}\eta_r \mathbf{F}_{kr}\mathbf{B}\Phi^1 + \sum_{r=1}^{q}\mathbf{F}_{kr}\mathbf{Q}_r^*, \quad (k = 1, 2, ..., q). \tag{5.17}$$

To obtain control force $Q(t)$, we have to first solve the adjoint system (3.9) in Chapter 3, by similar Galerkin resudual technique. As in the preceding steps, invoking the same there will be matrix equation similar to (5.13) for the the adjoint system (3.9) and eventually it conducts the scheme

$$\mathbf{Y}_k = -\sum_{r=1}^{q}\xi_r \mathbf{F}_{kr}\mathbf{B}\Theta^0 - \sum_{r=1}^{q}\eta_r \mathbf{F}_{kr}\mathbf{B}\Theta^1, \quad (k = 1, 2, ..., q) \tag{5.18}$$

as $Q(t)$ is absent there. The vectors $\Theta^0 = [\theta_i(0)]_{p\times 1}$, $\Theta^1 = [\dot{\theta}_i(0)]_{p\times 1}$ and $\mathbf{Y}_k$ $(k = 1, 2, ..., q)$ in (5.18) are the corresponding vectors to Φ^0, Φ^1 in the form (5.6) and $\mathbf{X}_k$ $(k = 1, 2, ..., q)$ in (5.12) respectively for the adjoint system (3.9). The above relation (5.18) can be rewritten as

$$\mathbf{Y}_k = \mathbf{L}_k^0\Theta^0 + \mathbf{L}_k^1\Theta^1, \quad (k = 1, 2, ..., q) \tag{5.19}$$

where $\mathbf{L}_k^0$ and $\mathbf{L}_k^1$ $(k = 1, 2, ..., q)$ are $p \times p$ square marices. To this point, knowing the vectors Θ^0 and Θ^1, each vector $\mathbf{Y}_k$ $(k = 1, 2, ..., q)$ can be computed from the relation (5.19).

On the other hand, as in the theory the control force $Q(t)$ is followed finally by the expression

$$Q(t) = \beta_0\theta(0, t), \tag{5.20}$$

with the help of (5.1), (5.2), (5.7) and (5.12), it can be written as

$$Q(t) = \beta_0\sum_{i=1}^{p}a_1^i\theta_i(t) = \beta_0\mathbf{I}_0\Theta(t) = \beta_0\Big[\sum_{k=1}^{q}\mathbf{I}_0\mathbf{Y}_k\Big(\frac{t}{T}\Big)^{k+1} + \mathbf{I}_0\Theta^0 + t\mathbf{I}_0\Theta^1\Big], \tag{5.21}$$

where the vector $\Theta(t)$ is the corresponding vector of $\Phi(t)$ in (5.7) for the adjoint system (3.9) and $\mathbf{I}_0 = [a_1^i]_{1\times p}$ is a row vector. Utilizing the relation (5.19), the control force $Q(t)$ can be reduced in terms of Θ^0 and Θ^1, as

$$Q(t) = \beta_0\Big[\sum_{K=1}^{q}(\mathbf{I}_0\mathbf{L}_k^0\Theta^0 + \mathbf{I}_0\mathbf{L}_k^1\Theta^1)\Big(\frac{t}{T}\Big)^{k+1} + \mathbf{I}_0\Theta^0 + t\mathbf{I}_0\Theta^1\Big] \tag{5.22}$$

Again plugging this $Q(t)$ in (5.15), we have

$$\mathbf{Q}_r^* = \mathbf{K}_r^0\Theta^0 + \mathbf{K}_r^1\Theta^1, \quad (r = 1, 2, ..., q) \tag{5.23}$$

where the explicit form of the square matrices $\mathbf{K}_r^0$ and $\mathbf{K}_r^1$ $(r = 1, 2, ...q)$ are given by

$$\mathbf{K}_r^0 = \lambda\beta_0\Big[\sum_{k=1}^{q}\Big(\frac{k(k+1)}{T(r+k+1)}\mathbf{C}\mathbf{I}_0 + \frac{T}{r+k+3}\mathbf{D}\mathbf{I}_0\Big)\mathbf{L}_k^0 + \frac{T}{r+2}\mathbf{D}\mathbf{I}_0\Big]$$
$$\mathbf{K}_r^1 = \lambda\beta_0\Big[\sum_{k=1}^{q}\Big(\frac{k(k+1)}{T(r+k+1)}\mathbf{C}\mathbf{I}_0 + \frac{T}{r+k+3}\mathbf{D}\mathbf{I}_0\Big)\mathbf{L}_k^1 + \frac{T^2}{r+3}\mathbf{D}\mathbf{I}_0\Big] \quad (5.24)$$

Now we substitute $\mathbf{Q}_r^*$ from (5.23) into (5.17), which clearly yields the $p \times p$ matrices $\mathbf{G}_k^0$, $\mathbf{G}_k^1$, $\mathbf{H}_k^0$, $\mathbf{H}_k^1$ for each $k = 1, 2, ..., q$ such that

$$\mathbf{X}_k = \mathbf{G}_k^0\Phi^0 + \mathbf{G}_k^1\Phi^1 + \mathbf{H}_k^0\Theta^0 + \mathbf{H}_k^1\Theta^1. \quad (5.25)$$

In this stage, we are now concerned on the time backward system (3.13), in Chapter 3. On the applicaton of similar Galerkin's residual technique with weight f_i over the interval $[0, l]$ to final conditions $\phi(x, T) = 0$, $(\partial\phi/\partial t)(x, T) = 0$, yields

$$\mathbf{A}^*\Phi(T) = \mathbf{C}^*\Big[\alpha\mathbf{I}_0\ddot{\Phi}(T) + \lambda Q(T)\Big]$$
$$\mathbf{A}^*\dot{\Phi}(T) = \mathbf{C}^*\Big[\alpha\mathbf{I}_0\dddot{\Phi}(T) + \lambda\dot{Q}(T)\Big] \quad (5.26)$$

in view of (5.1)–(5.3), where $\mathbf{A}^*$ is a $p \times p$ square matrix corresponds to $\mathbf{A}$ in (5.10) without the second term and the factor m. Similarly $\mathbf{C}^*$ is a $p \times 1$ column vector corresponds to $\mathbf{C}$ in (5.11) without the the factor m. With the help of the relations (5.7), (5.12), and (5.21), we have from (5.26)

$$\sum_{k=1}^{q}\mathbf{A}^*\mathbf{X}_k + \mathbf{A}^*\Phi^0 + T\mathbf{A}^*\Phi^1 = \mathbf{C}^*\Big[\frac{\alpha}{T^2}\sum_{k=1}^{q}k(k+1)\mathbf{I}_0\mathbf{X}_k + \lambda\beta_0\Big(\sum_{k=1}^{q}\mathbf{I}_0\mathbf{Y}_k + \mathbf{I}_0\Theta^0 + T\mathbf{I}_0\Theta^1\Big)\Big]$$
$$\frac{1}{T}\sum_{k=1}^{q}(k+1)\mathbf{A}^*\mathbf{X}_k + \mathbf{A}^*\Phi^1 = \mathbf{C}^*\Big[\frac{\alpha}{T^3}\sum_{k=2}^{q}(k-1)k(k+1)\mathbf{I}_0\mathbf{X}_k + \lambda\beta_0\Big(\frac{1}{T}\sum_{k=1}^{q}(k+1)\mathbf{I}_0\mathbf{Y}_k + \mathbf{I}_0\Theta^1\Big)\Big]$$

which on the use of (5.19) and (5.25) ultimately leads to the matrix equations of the form

$$\mathbf{P}\Theta^0 + \mathbf{Q}\Theta^1 = \mathbf{R}\Phi^0 + \mathbf{S}\Phi^1$$
$$\mathbf{U}\Theta^0 + \mathbf{V}\Theta^1 = \mathbf{W}\Phi^0 + \mathbf{Z}\Phi^1 \quad (5.27)$$

where the explicit form of all $p \times p$ square matrices $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$, $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, $\mathbf{Z}$ are as follows

$$
\begin{aligned}
\mathbf{P} &= \sum_{k=1}^{q}\mathbf{A}^*\mathbf{H}_k^0 - \frac{\alpha}{T^2}\mathbf{C}^*\mathbf{I}_0\sum_{k=1}^{q}k(k+1)\mathbf{H}_k^0 - \lambda\beta_0\mathbf{C}^*\mathbf{I}_0\sum_{k=1}^{q}\mathbf{L}_k^0 - \lambda\beta_0\mathbf{C}^*\mathbf{I}_0 \\
\mathbf{Q} &= \sum_{k=1}^{q}\mathbf{A}^*\mathbf{H}_k^1 - \frac{\alpha}{T^2}\mathbf{C}^*\mathbf{I}_0\sum_{k=1}^{q}k(k+1)\mathbf{H}_k^1 - \lambda\beta_0\mathbf{C}^*\mathbf{I}_0\sum_{k=1}^{q}\mathbf{L}_k^1 - \lambda\beta_0 T\mathbf{C}^*\mathbf{I}_0 \\
\mathbf{R} &= \frac{\alpha}{T^2}\mathbf{C}^*\mathbf{I}_0\sum_{k=1}^{q}k(k+1)\mathbf{G}_k^0 - \sum_{k=1}^{q}\mathbf{A}^*\mathbf{G}_k^0 - \mathbf{A}^* \\
\mathbf{S} &= \frac{\alpha}{T^2}\mathbf{C}^*\mathbf{I}_0\sum_{k=1}^{q}k(k+1)\mathbf{G}_k^1 - \sum_{k=1}^{q}\mathbf{A}^*\mathbf{G}_k^1 - T\mathbf{A}^* \\
\mathbf{U} &= \frac{1}{T}\sum_{k=1}^{q}(k+1)\mathbf{A}^*\mathbf{H}_k^0 - \frac{\alpha}{T^3}\mathbf{C}^*\mathbf{I}_0\sum_{k=1}^{q}(k-1)k(k+1)\mathbf{H}_k^0 - \frac{\lambda\beta_0}{T}\mathbf{C}^*\mathbf{I}_0\sum_{k=1}^{q}(k+1)\mathbf{L}_k^0 \\
\mathbf{V} &= \frac{1}{T}\sum_{k=1}^{q}(k+1)\mathbf{A}^*\mathbf{H}_k^1 - \frac{\alpha}{T^3}\mathbf{C}^*\mathbf{I}_0\sum_{k=1}^{q}(k-1)k(k+1)\mathbf{H}_k^1 - \frac{\lambda\beta_0}{T}\mathbf{C}^*\mathbf{I}_0\sum_{k=1}^{q}(k+1)\mathbf{L}_k^1 - \lambda\beta_0\mathbf{C}^*\mathbf{I}_0 \\
\mathbf{W} &= \frac{\alpha}{T^3}\mathbf{C}^*\mathbf{I}_0\sum_{k=1}^{q}(k-1)k(k+1)\mathbf{G}_k^0 - \frac{1}{T}\sum_{k=1}^{q}(k+1)\mathbf{A}^*\mathbf{G}_k^0 \\
\mathbf{Z} &= \frac{\alpha}{T^3}\mathbf{C}^*\mathbf{I}_0\sum_{k=1}^{q}(k-1)k(k+1)\mathbf{G}_k^1 - \frac{1}{T}\sum_{k=1}^{q}(k+1)\mathbf{A}^*\mathbf{G}_k^1 - \mathbf{A}^*
\end{aligned}
\tag{5.28}
$$

Now we can solve the vectors Θ^0 and Θ^1 from (5.27) with the help of the (5.6) following given initial conditions (3.7) of the flexural vibration problem (3.8) in Chapter 3. After knowing the vectors Θ^0 and Θ^1, the control force $Q(t)$ at time t can be obtained approximately from (5.22) and the vecor $\mathbf{Q}_r^*$ $(r = 1, 2, ..., q)$ from (5.23). Hence the vector $\mathbf{X}_k$ $(k = 1, 2, .., q)$ can be computed in view of (5.17) which at once yields the vector $\Psi(t)$ from (5.12) and then the vector $\Phi(t)$ at time t by the relation (5.7). Hence finally, we can compute the approximate shape function in virtue of (5.1). Since the scheme is direct and usually low values of p and q are needed, computation proceeds very fast.

5.3 Approximate Numerical Result

In practical procedure, we may assume that $y_0(x)$ and $y_1(x)$ are approximated by suitable polynomials (by measurement at suitable discrete points along the length of the panel) satisfying the corresponding homogeneous boundary conditions (3.5)–(3.6).

The functions thus become candidates for f_i and we can take

$$f_1(x) = y_0(x), \qquad f_2(x) = y_1(x) \tag{5.29}$$

so that

$$\phi_1(0) = 1 \quad \text{and} \quad \phi_i(0) = 0, \quad (i = 2, 3, ..., p), \tag{5.30}$$

$$\dot{\phi}_2(0) = 1 \quad \text{and} \quad \dot{\phi}_i(0) = 0, \quad (i = 1, 3, 4, ..., p), \tag{5.31}$$

In addition to these we can take another function $f_3(x)$ which is a simple monotonic function, since with increasing time the equilibrium position $y = 0$ is approached. Thus we can take

$$f_3(x) = \frac{x^2}{l^2} - \frac{1}{2}\frac{x^4}{l^4} + \frac{1}{5}\frac{x^5}{l^5}. \tag{5.32}$$

The last function $f_4(x)$ (with $p = 3$) is similarly taken as

$$f_4(x) = -\frac{1}{4}\frac{x^2}{l^2} + \frac{1}{6}\frac{x^3}{l^3} - \frac{1}{24}\frac{x^4}{l^4} \tag{5.33}$$

satisfying the non-homogeneous boundary condition (3.4).

The model parameters for numerical computation for the control problem are chosen as follows (in MKS units):

Length of the panel $l = 3.6$ meter,

Mass per unit length of the panel $m = 5.9$ kilogram/meter,

Poisson ratio $\nu = 0.33$, Rigidity $D = 6.9$ kilogram $\times$ meter ^{3}/sec ^{2},

Mass of the hub $m_h = 12.2$ kg. $\beta_0 = 1$.

For this panel $T_0 = 15.26$ sec. We consider two examples of initial conditions. In the first we take

$$y_0(x) = -\frac{1}{200} + \frac{x^2}{l^2} - 15\frac{x^4}{l^4} + \frac{67}{2}\frac{x^5}{l^5} - 27\frac{x^6}{l^6} + \frac{53}{7}\frac{x^7}{l^7}, \quad y_1(x) = 0, \tag{5.34}$$

in which $y_0(x)$ has a wavy shape and in the second, we take

$$y_0(x) = \frac{1}{100}\Big(1 + \frac{x^2}{l^2} + 2\frac{x^4}{l^4} - \frac{14}{5}\frac{x^5}{l^5} + \frac{x^6}{l^6}\Big), \quad y_1(x) = \frac{1}{5}\Big(\frac{x^4}{l^4} - \frac{6}{5}\frac{x^5}{l^5} + \frac{2}{5}\frac{x^6}{l^6}\Big), \tag{5.35}$$

where a monotonic velocity is imparted with a small monotonic displacement. Applying the above computational scheme with $p = 2, q = 4$ and $p = 3, q = 3$ respectively it is observed that the Galerkin approximation yields increasingly better results for T higher than 15.26. The above results for the dynamic deflection and the control force for the first example with $T = 20$ sec. are presented in the Figures 5.1 and 5.2, while those for the second example with $T = 30$ sec. are presented in Figures 5.3 and 5.4

respectively. It has been observed that velocities are lower by an order of magnitude. Finally, we remark that for very accurate results even for low values of T (higher than T_0) we may need a full, space-time Galerkin finite element technique. Such a technique will however need greater computational time.

5.4 Concluding Remarks

Here we have constructed a closed form numercal scheme for the problem of exact controllability of tranverse vibrations of a flexible panel attached to a rigid hub at one end and totally free at the other to obtain an approximate closed form solution together with approximate boundary control. The treatement of the problem is based on Galerkin's residual technique. The scheme works fast as illustrated in the two examples. In practical application since the theory is exact, the parameters involved should be known as accurately as possible. The initial displacement and velocity $y_0(x)$ and $y_1(x)$ when sufficiently smooth need be measured at a limited number of points along the length of the panel and approximated by polynomial functions.

Of the other parameters required in the theory D may be determined from some dynamical test, while m, m_h and l can be ascertained quite accurately. Nevertheless, approximations and uncertainties in measurements do pose the question of robustness (cf. [18]) of the exact theory and may be addressed to theoretically. We may however note that dissipation of energy takes place in actual systems with significant material damping in the panel, and frictional and other losses in the hub rendering the system asymptotically stable. The level of performances should thus be good under these cercumstances. In the earlier literature (cf. [22,23,24]) using modal decomposition followed by finite state representation, we may note that the uncertainities are introduced as Gaussian white noise followed by Kalman filtering.

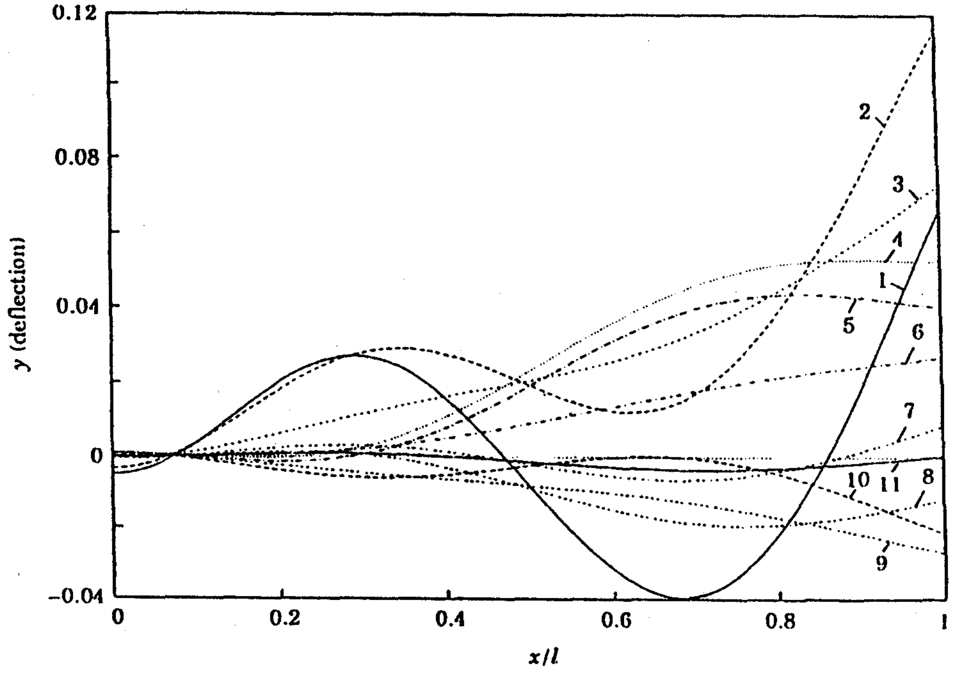

Figure 5.1. *The approximate deflections of the panel along the length with different time for* $T = 20$. *Numbers* $1 - 11$ *respectively are the approximate positions of the panel at the times* $t = 0$, $t = 0.1T$, $t = 0.2T$, $t = 0.3T$, $t = 0.4T$, $t = 0.5T$, $t = 0.6T$, $t = 0.7T$, $t = 0.8T$, $t = 0.9T$, $t = T$.

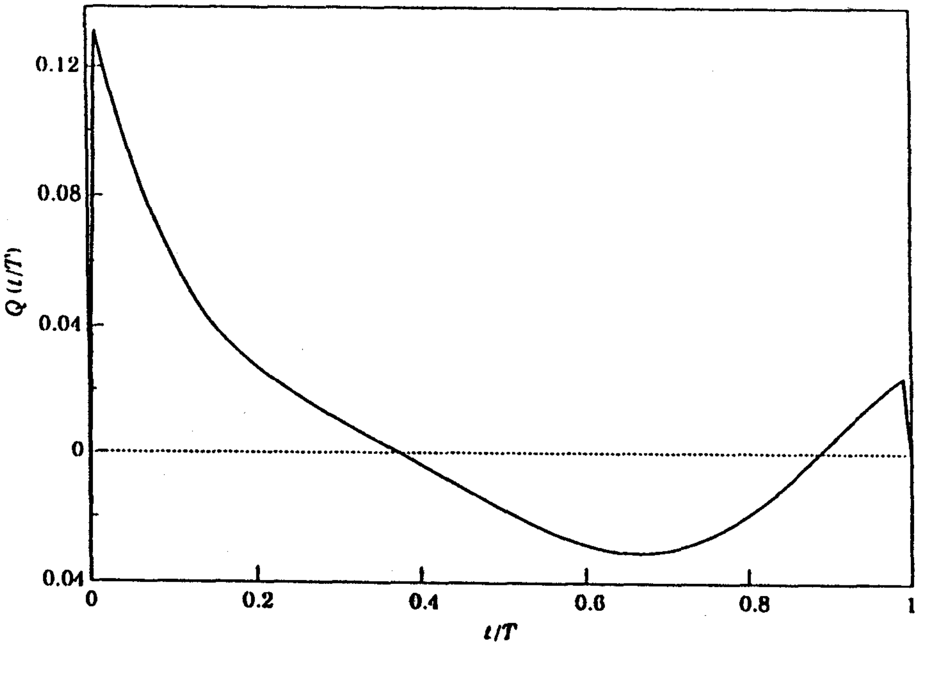

Figure 5.2. ***The approximate response of the control force with time for*** $T = 20$. ***Note that application and removal of the control force at initial and final times is sudden.***

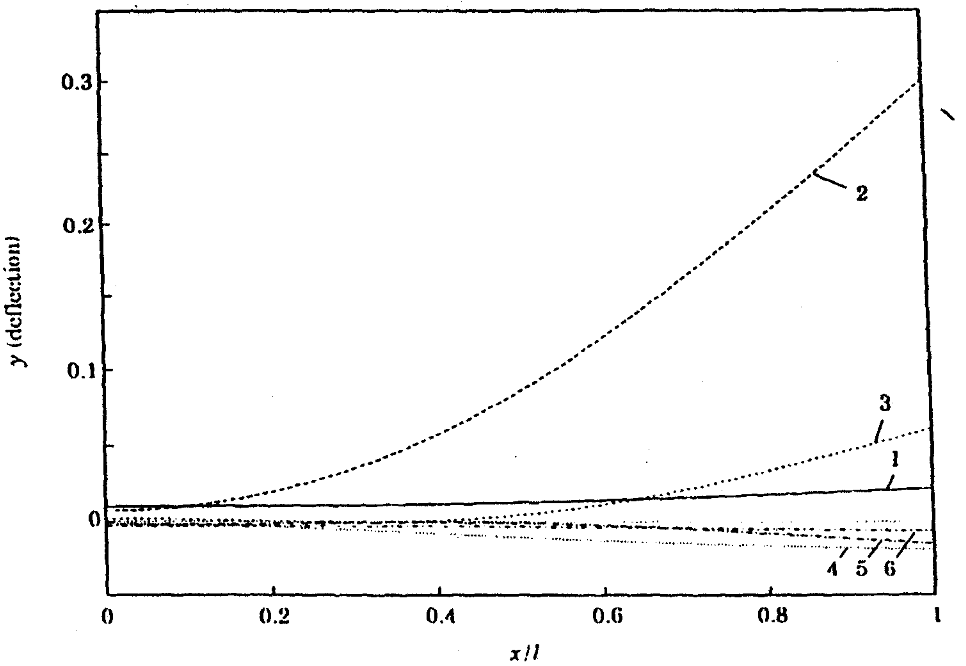

Figure 5.3. *The approximate deflections of the panel along the length with different time for $T = 30$. Numbers $1-6$ respectively are the approximate positions of the panel at the times $t = 0$, $t = 0.2T$, $t = 0.4T$, $t = 0.6T$, $t = 0.8T$, $t = T$.*

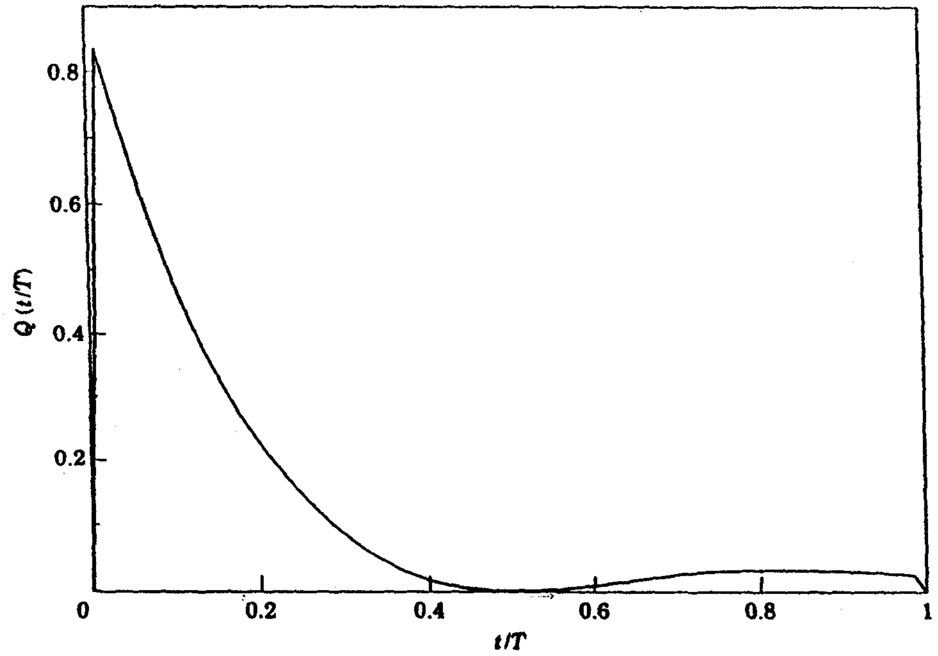

Figure 5.4. *The approximate response of the control force with time for $T = 30$. Note that application and removal of the control force at initial and final times is sudden.*

Part II – UNIFORM STABILIZATION

CHAPTER 6

BOUNDARY STABILIZATION OF TORSIONAL VIBRATIONS OF A FLEXIBLE PANEL

6.1 Introduction and Mathematical Formulation

Research on the topic of boundary stabilization for distributed parameter systems governed by wave equation have been developing with great importance during the last few decades. We shall be dealing in this Chapter, boundary stability for the solution of the hybrid dynamical model of torsional vibrations of the uniform rectangular panel as formulated in Chapter 1. In the mathematical literature, stabilization (strongly) of a system means the convergence of the solution of the system to zero as time tends to infinity. If corresponding to all initial data with finite initial energy, the solution of the system converges uniformly with introduction of a boundary stabilizer or damping device, it is called a uniform boundary stable system.

Though the study of stabilization for the solution of wave equation started in the early sixties (cf. Lax *et al.* [50], Morawetz [63]), Chen [6] first obtained the explicit form of exponential energy decay rate (uniform boundary stabilization) for the same in a *star-complemented strongly star-shaped* domain (SCSSSD) in $\mathbf{R}^n$. The results of uniform stabilization by means of exponential energy decay estimate have been later improved by Lagnese [43], Komornik [37] by constructing speacial type of feedback boundary dampings and obtained somewhat faster energy decay rates. The stabilization of the solution of hybrid vibrating system has been described throughly by Littman and Markus [55] and Rao [72,73], with a boundary feedback applied at the end with a lumped mass. All the above investigations have shown the stability of wave equation or Euler-Bernoulli beam equation, clamped at one end and feedback damping or control force applied on the other end. But for the class of systems such as solar cell array, robot with flexible links or spacecraft with flexible appendages, it is practically undesirable or the most difficult task to apply boundary damping or stabilizer on at the free end of the structure where as to apply it on the other end is much easier in order to obtain a good performance of the overall system.

Thus to study boundary stabilization of the solution of torsional vibrations of the uniform rectangular panel as describe in Chapter 1 by the mathematical problem (1.7), we need a boundary stabilizer such that the solution of the system corresponding to initial data with finite energy decays exponentially in the energy space as time $t \to +\infty$. For this, we now select a viscous boundary damping force (in fact, a passive damping force) applicable on the rigid hub of the panel, that means, $Q(t)$ in (1.4) or in the system (1.7) is taken as proportional to $\dot{\phi}(0,t)$, say, $Q(t) = b\dot{\phi}(0,t)$ where b is a positive constant. That means, we select a boundary velocity feedback damping to describe the asymptotic behavior of the the system (1.7). Hence to study the boundary stabilization of the torsional vibration problem, the following mathematical system of equations is to be concerned.

$$\begin{aligned} &\ddot{\phi}(x,t) = c^2\phi''(x,t) && && 0 \le x \le l,\ t \ge 0, \\ &\phi(x,0) = \phi_0(x), && \dot{\phi}(x,0) = \phi_1(x), && 0 \le x \le l, \\ &\phi'(0,t) = \alpha\,\ddot{\phi}(0,t) + \lambda b\dot{\phi}(0,t) \quad \text{and} && \phi'(l,t) = 0, && t \ge 0, \end{aligned} \tag{6.1}$$

where the parameters c, α, λ remain same as considered in Chapter 1, and prime and dot denote differentiations with respect to x and t respectively.

6.2 Energy of the System

Associated with each solution of (6.1), the total energy at time t is defined by

$$E(t) = \frac{1}{2}\int_0^l (\dot{\phi}^2 + c^2\phi'^2)dx + \frac{1}{2}c^2\alpha\dot{\phi}^2(0,t). \tag{6.2}$$

Differentiating (6.2) with respect to t and using the first equation of (6.1), we have

$$\dot{E}(t) = c^2\int_0^l \frac{\partial}{\partial x}(\dot{\phi}\phi')dx + c^2\alpha\dot{\phi}(0,t)\,\ddot{\phi}(0,t).$$

Integrating by parts and applying the boundary conditions of (6.1), the above yields

$$\dot{E}(t) = -c^2\lambda b\dot{\phi}^2(0,t) \le 0. \tag{6.3}$$

The negativity of the right hand side of (6.3) shows that the energy $E(t)$ of the system (6.1) is nonincreasing with time and the system is thus energy dissipating due to viscous boundary damping at the hub end. Evidently, it follows from (6.3) that

$$E(t) \le E(0), \qquad \text{for } t \ge 0. \tag{6.4}$$

6.3 Uniform Boundary Stability Result

As the energy of the system decays with time, so naturally the question arises on the conditions for which it decays uniformly. Owing to this fact, the main interest in this Chapter is to obtain explicitly the uniform exponential energy decay estimate for the solution of the vibrating system (6.1), that means we want to establish the result of the form

$$E(t) \leq Me^{-\beta t}E(0), \qquad t \geq 0 \tag{6.5}$$

for some reals $\beta > 0$ and $M \geq 1$.

The result (6.5), for the system (6.1) is ensured from the Theorem:

Theorem 6.1. Let $\phi(x,t)$ is a solution of (6.1) corresponding to initial conditions $\phi_0 \in H^2[0,l]$ and $\phi_1 \in H^1[0,l]$. Then the energy of the system (6.1) defined by (6.2) decays exponentially with time, i.e., $E(t)$ satisfies (6.5) for some reals $\beta > 0$ and $M \geq 1$.

Before proving Theorem 6.1, we first consider the followings.

Lemma 6.1. If $\phi(x,t)$ is a solution of (6.1) then the function $\rho(t)$ defind by

$$\rho(t) = \frac{1}{2}\int_0^l (\dot{\phi}'^2 + c^2\phi''^2)dx + \frac{\alpha}{2}\ddot{\phi}^2(0,t) \tag{6.6}$$

is nonincreasing with time.

Proof. Differentiating (6.6) with respect to t, we have

$$\dot{\rho}(t) = \int_0^l (\dot{\phi}'\ddot{\phi}' + c^2\phi''\dot{\phi}'')dx + \alpha\ddot{\phi}(0,t)\dddot{\phi}(0,t).$$

Introducing the first equation of (6.1) and then integrating by parts, the above leads to

$$\dot{\rho}(t) = \left[\dot{\phi}'\ddot{\phi}\right]_0^l + \alpha\ddot{\phi}(0,t)\dddot{\phi}(0,t).$$

Applying the boundary conditions of (6.1), we get

$$\dot{\rho}(t) = -\lambda b\ddot{\phi}^2(0,t) \leq 0. \tag{6.7}$$

From the negativity of the right hand side of (6.7), the Lemma 6.1 follows at once.

Lemma 6.2. If we define a function $\rho_0(t)$ by

$$\rho_0(t) = \int_0^l (x-l)\dot{\phi}\phi' dx \tag{6.8}$$

then

$$\dot{\rho}_0(t) \leq \frac{1}{2}(l + c^2\alpha + 2c^2\lambda^2b^2l)\dot{\phi}^2(0,t) + c^2\alpha^2 l\ddot{\phi}^2(0,t) - E(t). \tag{6.9}$$

Proof. Differentiating (6.8) with respect to t and using the governing equation of (6.1), we have

$$\dot{\rho}_0(t) = \int_0^l (x-l)\left(\dot{\phi}\dot{\phi}' + c^2\phi''\phi'\right)dx.$$

Integrating by parts, the above becomes

$$\dot{\rho}_0(t) = \frac{1}{2}\left[(x-l)(\dot{\phi}^2 + c^2\phi'^2)\right]_0^l - \frac{1}{2}\int_0^l (\dot{\phi}^2 + c^2\phi'^2)dx.$$

Applying the boundary conditions of (6.1), we thus obtain

$$\dot{\rho}_0(t) = \frac{1}{2}l\left(\dot{\phi}^2(0,t) + c^2\phi'^2(0,t)\right) + \frac{c^2\alpha}{2}\dot{\phi}^2(0,t) - E(t) \tag{6.10}$$

by the energy equation (6.2). Since $\phi'(0,t) = \alpha\ddot{\phi}(0,t) + \lambda b\dot{\phi}(0,t)$, we can write

$$\phi'^2(0,t) = \left(\alpha\ddot{\phi}(0,t) + \lambda b\dot{\phi}(0,t)\right)^2 \leq 2\left[\alpha^2\ddot{\phi}^2(0,t) + \lambda^2 b^2\dot{\phi}^2(0,t)\right]. \tag{6.11}$$

Now plugging (6.11) into (6.10), the Lemma 6.2 follows immediately.

At this stage we are ready to prove the main Theorem.

Proof of Theorem. We now construct a function $E_\beta(t)$ by

$$E_\beta(t) = \left[E(t) + 2\beta\left(\rho_0(t) + \frac{c^2\alpha^2 l}{\lambda b}\rho(t)\right)\right]e^{\beta t}, \qquad \text{for } t \geq 0, \tag{6.12}$$

where $\beta > 0$ is a fixed constant defined later. We note that

$$|\rho_0(t)| = \left|\int_0^l (x-l)\dot{\phi}\phi' dx\right| \leq \frac{l}{2c}\int_0^l (\dot{\phi}^2 + c^2\phi'^2)dx \leq \frac{l}{c}E(t). \tag{6.13}$$

Since $E(t)$ and $\rho(t)$ are nonincreasing with time, therefore for every $\phi_0 \in H^2[0,l]$ and $\phi_1 \in H^1[0,l]$ we have $E(t) \leq E(0) < \infty$ and $\rho(t) \leq \rho(0) < \infty$. Hence there exists a positive constant δ such that

$$\rho(t) \leq \delta E(t). \tag{6.14}$$

We can assert that δ is bounded above for $t \geq 0$. This follows from the fact that by final value theorem of Laplace transform, for infinitely large values of t, $\rho(t)$ decays faster than $E(t)$, according to the expressions for $\dot{\rho}(t)$ and $\dot{E}(t)$ in (6.7) and (6.3) respectively.

Inserting the relations (6.13) and (6.14) into (6.12), we have

$$e^{\beta t}(1 - 2\beta\frac{l}{c})E(t) \leq E_\beta(t) \leq e^{\beta t}\left[1 + 2\beta\left(\frac{l}{c} + \frac{\delta c^2\alpha^2 l}{\lambda b}\right)\right]E(t). \tag{6.15}$$

Now we differentiate (6.12) with respect to t to obtain

$$\dot{E}_\beta(t) = e^{\beta t}\Big[\dot{E}(t) + 2\beta\big(\dot{\rho}_0(t) + \frac{c^2\alpha^2 l}{\lambda b}\dot{\rho}(t)\big)\Big] + \beta E_\beta(t). \tag{6.16}$$

Introducing the right inequality in (6.15), the relation (6.16) can be written as

$$\dot{E}_\beta(t) \le e^{\beta t}\Big[\dot{E}(t) + 2\beta\big(\dot{\rho}_0(t) + \frac{c^2\alpha^2 l}{\lambda b}\dot{\rho}(t)\big) + \beta\Big(1 + 2\beta\big(\frac{l}{c} + \frac{\delta c^2\alpha^2 l}{\lambda b}\big)\Big)E(t)\Big]. \tag{6.17}$$

Applying Lemma 6.2 and the relations (6.3), (6.7), the above yields

$$\dot{E}_\beta(t) \le e^{\beta t}\Big[\big[\beta(l + c^2\alpha + 2c^2\lambda^2 b^2 l) - c^2\lambda b\big]\dot{\phi}^2(0,t) + \beta\big[2\beta l\big(\frac{1}{c} + \frac{\delta c^2\alpha^2}{\lambda b}\big) - 1\big]E(t)\Big]. \tag{6.18}$$

Now we choose the constant $\beta > 0$ as

$$\beta = \min\left\{\frac{c^2\lambda b}{l + c^2\alpha + 2c^2\lambda^2 b^2 l}, \frac{1}{2l(1/c + \delta c^2\alpha^2/\lambda b)}\right\}. \tag{6.19}$$

Then we have from (6.18),

$$\dot{E}_\beta(t) \le 0, \qquad \text{for } t \ge 0 \tag{6.20}$$

and at the same time from (6.15),

$$e^{\beta t}\frac{2\beta l\delta c^2\alpha^2}{\lambda b}E(t) \le E_\beta(t) \le e^{\beta t}\big[1 + 2\beta l\big(\frac{1}{c} + \frac{\delta c^2\alpha^2}{\lambda b}\big)\big]E(t), \qquad \text{for } t \ge 0. \tag{6.21}$$

Integrating over 0 to t, it follows from (6.20),

$$E_\beta(t) \le E_\beta(0), \qquad \text{for } \ t \ge 0. \tag{6.22}$$

With the help of the inequalities in (6.21), it follows finally from (6.22) that

$$E(t) \le Me^{-\beta t}E(0), \qquad \text{for } \ t \ge 0 \tag{6.23}$$

where,

$$M = \frac{1 + 2\beta l(1/c + \delta c^2\alpha^2/\lambda b)}{2\beta l\delta c^2\alpha^2/\lambda b} \ge 1. \tag{6.24}$$

Which completes the proof.

6.4 Concluding Remarks

From (6.23) it follows that the uniform exponential energy decay rate β of vibrations will be maximum for largest admissible value of β given by the smaller value between the two in the relation (6.19). In this context, it should be remarked from (6.19) that the energy decay rate β is varing inversely with the length l of the panel. More precisely, vibrations of longer panel will take much more time to stabilize, which is highly singnificant for our problem because one end of the panel is totally free.

CHAPTER 7

BOUNDARY STABILIZATION OF TORSIONAL VIBRATIONS OF AN INTERNALLY DAMPED FLEXIBLE PANEL†

7.1 Introduction and Mathematical Formulation

In this Chapter, we study the uniform boundary stability for the solution of internally damped torsional vibrations of the hybrid system, consisting of a uniform rectangular panel with a rigid hub, at one end as described by system (2.3) in Chapter 2. In other words, we are concerned about the uniform stability of the problem of the previous Chapter incorporating internally damping of the material of the panel. In the literature, boundary stabilization deals with the existence of a boundary stabilizer such that each solution corresponding to initial data with finite energy decays exponentially in the energy space as $t \to +\infty$. To describe the asymptotic behavior of the system (2.3) in Chapter 2, we apply boundary control (stabilizer) $Q(t)$ in the system (2.3) as $b\dot{\phi}(0,t)$, where $b > 0$ is a finite real. In mathematical literature, this stabilizer is nothing but the viscous boundary damping force on the rigid hub of the panel.

Hence for the boundary stability of the problem of internally damped torsional vibrations is mathematically concerned by the following system.

$$\begin{aligned} &\ddot{\phi}(x,t) = c^2\phi''(x,t) + \mu\dot{\phi}''(x,t), && 0 \le x \le l, \quad t \ge 0, \\ &\phi(x,0) = \phi_0(x), \qquad \dot{\phi}(x,0) = \phi_1(x), && 0 \le x \le l, \\ &\phi'(0,t) = \alpha\,\ddot{\phi}(0,t) + \lambda b\dot{\phi}(0,t) \quad \text{and} \quad \phi'(l,t) = 0, && t \ge 0. \end{aligned} \tag{7.1}$$

Thus we are concerned about the asymptotic behavior in the presence of both internal material damping of the panel as well as boundary viscous damping on the rigid hub. The parameters c, α, λ are given by in Chapter 1.

†The contents of this chapter have been published in the paper *Exact Controllability and Boundary Stabilization of Torsional Vibrations of an Internally Damped Flexible Space Structure* —Gorain and Bose, 'Journal of Optimization Theory and Applications' Vol. 99, 423–442, (1998).

7.2 Energy of the System

For each solution of (7.1), the total energy at time t is defined by

$$E(t) = \frac{1}{2}\int_0^l (\dot{\phi}^2 + c^2\phi'^2)dx + \frac{1}{2}(c^2\alpha + \mu\lambda b)\dot{\phi}^2(0,t). \tag{7.2}$$

Differentiating (7.2) with respect to t and using the first equation of (7.1), we have

$$\dot{E}(t) = c^2\int_0^l \frac{\partial}{\partial x}(\dot{\phi}\phi')dx + \mu\int_0^l \dot{\phi}\dot{\phi}''dx + (c^2\alpha + \mu\lambda b)\dot{\phi}(0,t)\ddot{\phi}(0,t).$$

Integrating by parts and applying the boundary conditions of (7.1), yields

$$\dot{E}(t) = -c^2\lambda b\dot{\phi}^2(0,t) - \mu\alpha\dot{\phi}(0,t)\dddot{\phi}(0,t) - \mu\int_0^l \dot{\phi}'^2 dx. \tag{7.3}$$

The integral term on the right hand side of (7.3) shows that some energy is dissipating throughout the system due to incorporation of material damping of the panel. Similarly the first term on the right hand side of (7.3) shows energy dissipation due to boundary damping on the rigid hub of the panel. We now estimate $\dot{\phi}(0,t)\dddot{\phi}(0,t)$ as

$$\left|\dot{\phi}(0,t)\dddot{\phi}(0,t)\right| \le \frac{1}{2}\left[\frac{2c^2\lambda b}{\mu\alpha}\dot{\phi}^2(0,t) + \frac{\mu\alpha}{2c^2\lambda b}\dddot{\phi}^2(0,t)\right]. \tag{7.4}$$

If μ is sufficiently small compared to λ and b, then it follows from (7.3) that $\dot{E}(t) \le 0$, and $E(t)$ is nonincreasing with time, i.e.,

$$E(t) \le E(0), \qquad t \ge 0. \tag{7.5}$$

As our main interest in this section is to show explicitly the exponential energy decay rate of vibrations, that means we want to establish

$$E(t) \le Me^{-\beta t}E(0), \qquad t \ge 0 \tag{7.6}$$

for some positive β and some real $M \ge 1$, so the question is under what conditions the energy $E(t)$ satisfies (7.6).

7.3 Uniform Boundary Stability Result

Chen [6,9], Komornik [37] and Lagnese [42,43] have shown that if a bounded domain Ω in $\mathbf{R}^n$ has certain geometries, then the energy for the solution of second order wave equation with a viscous boundary feedback damping will decay uniformly exponentially. Chen [9] has also obtained faster energy decay rate of waves considering distributed viscous damping and boundary damping.

Exponential energy decay of vibrations for the problem governed by the set of equations (7.1) follows from the theorem:

Theorem 7.1. Let $\phi(x,t)$ is a solution of (7.1) corresponding to initial conditions $\phi_0 \in H^2[0,l]$ and $\phi_1 \in H^1[0,l]$ with $\mu > 0$ small enough. Then the energy of the system (7.1) defined by (7.2) decays exponentially as $t \to +\infty$, i.e., there exists constants $M \geq 1$ and $\beta > 0$ such that $E(t)$ satisfies (7.6).

Before proving Theorem 7.1, we first consider the following Lemma.

Lemma 7.1. If $\phi(x,t)$ is a solution of (7.1) then the function $\rho(t)$ defind by

$$\rho(t) = \frac{1}{2}\int_0^l (\dot{\phi}'^2 + c^2\phi''^2)dx + \frac{\alpha}{2}\ddot{\phi}^2(0,t) \tag{7.7}$$

is nonincreasing with time.

Proof. Differentiating (7.7) with respect to t and introducing the first equation of (7.1) we have

$$\dot{\rho}(t) = \int_0^l \left[\dot{\phi}'(c^2\phi''' + \mu\dot{\phi}''') + c^2\phi''\dot{\phi}''\right]dx + \alpha\ddot{\phi}(0,t)\dddot{\phi}(0,t).$$

Integrating by parts we obtain

$$\dot{\rho}(t) = \left[c^2\dot{\phi}'\phi'' + \mu\dot{\phi}'\dot{\phi}''\right]_0^l + \alpha\ddot{\phi}(0,t)\dddot{\phi}(0,t) - \mu\int_0^l \dot{\phi}''^2 dx.$$

Applying the boundary conditions of (7.1) we get

$$\dot{\rho}(t) = -\lambda b\ddot{\phi}^2(0,t) - \mu\int_0^l \dot{\phi}''^2 dx \leq 0. \tag{7.8}$$

Hence the Lemma follows.

Proof of Theorem. We now construct a function $E_\beta(t)$ by

$$E_\beta(t) = \left[E(t) + \mu\frac{l}{c}\rho(t) + \beta\int_0^l (2x\dot{\phi}\phi' + \mu\phi'^2)dx\right]e^{\beta t}, \qquad t \geq 0, \tag{7.9}$$

where $\beta > 0$ is a small fixed number satisfying

$$\beta \leq \frac{\mu}{2l^2}. \tag{7.10}$$

We note that

$$\left|\int_0^l 2x\dot{\phi}\phi' dx\right| \leq \frac{l}{c}\int_0^l (\dot{\phi}^2 + c^2\phi'^2)dx = \frac{2l}{c}E(t) \tag{7.11}$$

and

$$\int_0^l \phi'^2 dx \leq \frac{2}{c^2}E(t). \tag{7.12}$$

Using the results (7.11) and (7.12), we have from (7.9)

$$e^{\beta t}(1-2\beta\frac{l}{c})E(t) \le E_\beta(t) \le \Big[\Big(1+2\beta(\frac{l}{c}+\frac{\mu}{c^2})\Big)E(t)+\frac{\mu l}{c}\rho(t)\Big]e^{\beta t}. \tag{7.13}$$

Now differentiating (7.9) with respect to t and utilizing (7.1), we obtain after a simple calculation

$$\begin{aligned}\dot{E}_\beta(t) &= e^{\beta t}\Big[\dot{E}(t)+\frac{\mu l}{c}\dot{\rho}(t)+\beta\int_0^l x\frac{\partial}{\partial x}(\dot{\phi}^2+c^2\phi'^2)dx \\ &+2\mu\beta\int_0^l \phi'\frac{\partial}{\partial x}(x\dot{\phi}')dx\Big]+\beta E_\beta(t).\end{aligned} \tag{7.14}$$

Integrating by parts and introducing (7.2), we have from the above

$$\begin{aligned}\dot{E}_\beta(t) &= e^{\beta t}\Big[\dot{E}(t)+\frac{\mu l}{c}\dot{\rho}(t)+\beta\Big(l\dot{\phi}^2(l,t)+(c^2\alpha+\mu\lambda b)\dot{\phi}^2(0,t)\Big) \\ &-2\mu\beta\int_0^l x\dot{\phi}'\phi''dx-2\beta E(t)\Big]+\beta E_\beta(t).\end{aligned} \tag{7.15}$$

Applying the right inequality in (7.13), we obtain from (7.15)

$$\begin{aligned}\dot{E}_\beta(t) &\le e^{\beta t}\Big[\dot{E}(t)+\frac{\mu l}{c}\dot{\rho}(t)+\beta\Big(l\dot{\phi}^2(l,t)+(c^2\alpha+\mu\lambda b)\dot{\phi}^2(0,t)\Big) \\ &+\beta(\frac{2\beta l}{c}+\frac{2\mu\beta}{c^2}-1)E(t)+\mu\beta\frac{l}{c}\rho(t)-2\mu\beta\int_0^l x\dot{\phi}'\phi''dx\Big].\end{aligned} \tag{7.16}$$

Here we note that

$$\Big|\int_0^l x\dot{\phi}'\phi''dx\Big| \le \frac{l}{c}\rho(t). \tag{7.17}$$

Since $\rho(t)$ is nonincreasing as in Lemma 7.1, therefore we have $\rho(t)\le\rho(0)$. Hence for every $\phi_0\in H^2[0,l]$ and $\phi_1\in H^1[0,l]$, there exists a positive constant δ independent of μ such that

$$\rho(t)\le\delta E(t). \tag{7.18}$$

This follows from the fact that for infinitely large values of t, the leading terms of $\dot{\rho}(t)$ and $\dot{E}(t)$ in (7.8) and (7.3) for small μ are respectively $-\lambda b\ddot{\phi}^2(0,t)$ and $-c^2\lambda b\dot{\phi}^2(0,t)$. Hence for $t\to\infty$, $\rho(t)$ decays faster than $E(t)$ by the final value theorem of Laplace transform. Hence the assertion.

Introducing (7.10), (7.17) and (7.18) we therefore obtain from (7.16),

$$\begin{aligned}\dot{E}_\beta(t) &\le e^{\beta t}\Big[\dot{E}(t)+\frac{\mu l}{c}\dot{\rho}(t)+\frac{\mu}{2l^2}\Big(l\dot{\phi}^2(l,t)+(c^2\alpha+\mu\lambda b)\dot{\phi}^2(0,t)\Big) \\ &+\beta\Big(\mu\Big(\frac{\mu}{c^2l^2}+\frac{1}{cl}+\frac{3l\delta}{c}\Big)-1\Big)E(t)\Big]\end{aligned} \tag{7.19}$$

If we choose μ so small such that

$$\mu\Big(\frac{\mu}{c^2l^2}+\frac{1}{cl}+\frac{3l\delta}{c}\Big)\leq 1, \tag{7.20}$$

we have then from (7.19)

$$\begin{aligned}\dot{E}_\beta(t) \;\leq\; & e^{\beta t}\Big[\frac{\mu}{2l^2}\big[l\dot{\phi}^2(l,t)+(c^2\alpha+\mu\lambda b)\dot{\phi}^2(0,t)\big]-\mu\alpha\dot{\phi}(0,t)\dddot{\phi}(0,t) \\ & -\mu\frac{l}{c}\lambda b\ddot{\phi}^2(0,t)-\mu\int_0^l\dot{\phi}'^2dx-c^2\lambda b\dot{\phi}^2(0,t)\Big]\end{aligned} \tag{7.21}$$

by the use of (7.3) and (7.8).

Since

$$\dot{\phi}(l,t)=\dot{\phi}(0,t)+\int_0^l\dot{\phi}'dx,$$

therefore applying the inequalities

$$(a+b)^2\leq 2(a^2+b^2) \tag{7.22}$$

and

$$\Big(\int_0^l fgdx\Big)^2\leq\int_0^l f^2dx\int_0^l g^2dx, \tag{7.23}$$

it can be easily established that

$$\dot{\phi}^2(l,t)\leqq 2\Big[\dot{\phi}^2(0,t)+l\int_0^l\dot{\phi}'^2dx\Big]. \tag{7.24}$$

Introducing (7.24) into (7.21), we obtain

$$\begin{aligned}\dot{E}_\beta(t) \;\leq\; & e^{\beta t}\Big[\frac{\mu}{2l^2}\Big(2l+c^2\alpha+\mu\lambda b\Big)\dot{\phi}^2(0,t)-\mu\alpha\dot{\phi}(0,t)\dddot{\phi}(0,t) \\ & -\mu\frac{l\lambda b}{c}\ddot{\phi}^2(0,t)-c^2\lambda b\dot{\phi}^2(0,t)\Big]\end{aligned} \tag{7.25}$$

As μ is taken small enough, so the expression in the bracket on the right hand side of (7.25) can be made nonpositive for $\mu\leq\mu_0$. It should be remarked that this μ_0 depends on the viscous boundary damping parameter b. It then follows from (7.25) that

$$\dot{E}_\beta(t)\leq 0,\qquad t\geq 0. \tag{7.26}$$

Thus we have

$$E_\beta(t)\leq E_\beta(0),\qquad t\geq 0. \tag{7.27}$$

On the other hand using the inequalities (7.10) and (7.20), we have

$$1-2\beta\frac{l}{c}\geq\mu\Big(\frac{\mu}{c^2l^2}+\frac{3l\delta}{c}\Big). \tag{7.28}$$

Therefore the inequalities in (7.13) can be written as

$$e^{\beta t}\mu\Big(\frac{\mu}{c^2l^2}+\frac{3l\delta}{c}\Big)E(t)\leq E_\beta(t)\leq e^{\beta t}\Big[1+\mu\Big(\frac{\mu}{c^2l^2}+\frac{1}{cl}+\frac{l\delta}{c}\Big)\Big]E(t) \qquad (7.29)$$

by (7.10) and (7.18). With the help of (7.29) it follows finally from (7.27) that

$$E(t)\leq Me^{-\beta t}E(0), \qquad t\geq 0 \qquad (7.30)$$

where,

$$M=\frac{1+\mu\big(\mu/c^2l^2+1/cl+l\delta/c\big)}{\mu\big(\mu/c^2l^2+3l\delta/c\big)}. \qquad (7.31)$$

Hence the Theorem.

7.4 Concluding Remarks

It follows from (7.10) that uniform exponential energy decay rate β for the solution of internally damped torsional vibration problem governed by (7.1) explicitly depends on the internal damping parameter μ. As $\mu > 0$ is taken small satisfying the relations (7.20) and $\mu \leq \mu_0$, the exponential energy decay rate will be maximum for the selection of largest admissible value of μ. Again μ_0 depends on the boundary damping parameter b, it is concluded that the uniform exponential energy rate β implicitly depends on b also.

CHAPTER 8

BOUNDARY STABILIZATION OF FLEXURAL VIBRATIONS OF A FLEXIBLE PANEL†

8.1 Introduction and Mathematical Formulation

In the preceding two Chapters, we have discussed uniform boundary stabilization of undamped and internally damped torsional vibrations of a hybrid flexible space structure with a damping device at one end. We study in this Chapter, the uniform boundary stabilization of flexural vibrations of the same hybrid structure consisting of an Euler-Bernoulli panel with a rigid hub hoisted at one end as described in Chapter 3. The objective here, is to study the stability of the overall system by means of a uniform exponential energy decay estimate for the solution of the system under suitable stabilizing force $Q(t)$ applied only at the rigid hub.

To study uniform boundary stabilization, we assume that $Q(t)$ in (3.3) of Chapter 3, is proportional to $(\partial y/\partial t)(0,t)$ say, $Q(t) = b(\partial y/\partial t)(0,t)$ i.e., a viscous boundary damping stabilizer is present at the hub end, the constant $b > 0$ being the viscous damping parameter. Therefore, the mathematical problem to be persued for uniform boundary stabilization of the hybrid structure as described in Chapter 3, is governed by the following boundary value problem

$$\begin{aligned}
&m\frac{\partial^2 y}{\partial t^2}(x,t) + D\frac{\partial^4 y}{\partial x^4}(x,t) = 0, && 0 \le x \le l,\ t \ge 0,\\
&\frac{\partial^3 y}{\partial x^3}(0,t) + \alpha\frac{\partial^2 y}{\partial t^2}(0,t) + \lambda b\frac{\partial y}{\partial t}(0,t) = 0, \quad \frac{\partial y}{\partial x}(0,t) = 0, && t \ge 0,\\
&\frac{\partial^2 y}{\partial x^2}(l,t) = 0, \qquad \frac{\partial^3 y}{\partial x^3}(l,t) = 0, && t \ge 0,\\
&y(x,0) = y_0(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = y_1(x), && 0 \le x \le l,
\end{aligned} \tag{8.1}$$

where the parameters m, D, α, λ take the same values as those of in Chapter 3.

†The contents of this chapter have been communicated in the form a paper ***Boundary Stabilization of a Hybrid Euler-Bernoulli Beam*** —Gorain and Bose, 'Proceedings Indian Academy of Sciences (Mathematical Sciences)'.

8.2 Energy of the System

Associated with each solution of (8.1), the total energy at time t is defined by the functional

$$E(t) = \frac{1}{2}\int_0^l \Big[m\Big(\frac{\partial y}{\partial t}\Big)^2 + D\Big(\frac{\partial^2 y}{\partial x^2}\Big)^2\Big]dx + \frac{1}{2}m_h\Big(\frac{\partial y}{\partial t}(0,t)\Big)^2. \tag{8.2}$$

Now differentiating (8.2) with respect to t and replacing $m(\partial^2 y/\partial t^2)$ by $-D(\partial^4 y/\partial x^4)$, we obtain

$$\frac{dE}{dt} = D\int_0^l \frac{\partial}{\partial x}\Big(\frac{\partial^2 y}{\partial x \partial t}\frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial t}\frac{\partial^3 y}{\partial x^3}\Big)dx + m_h\frac{\partial y}{\partial t}(0,t)\frac{\partial^2 y}{\partial t^2}(0,t). \tag{8.3}$$

Applying the boundary conditions of (8.1), we get

$$\begin{aligned}\frac{dE}{dt} &= -D\Big[\alpha\frac{\partial^2 y}{\partial t^2}(0,t) + \lambda b\frac{\partial y}{\partial t}(0,t)\Big]\frac{\partial y}{\partial t}(0,t) + m_h\frac{\partial y}{\partial t}(0,t)\frac{\partial^2 y}{\partial t^2}(0,t) \\ &= -b\Big(\frac{\partial y}{\partial t}(0,t)\Big)^2 \le 0, \end{aligned} \tag{8.4}$$

for all $t \ge 0$, as $\alpha = m_h/D$, $\lambda = 1/D$ defined in Chapter 3. This implies

$$E(t) \le E(0) \qquad \text{for all} \quad t \ge 0. \tag{8.5}$$

Hence the energy $E(t)$ is nonincreasing with time and the system (8.1) is energy dissipating due to the incorporation of boundary stabilizer. As the energy decays, our main interest is to obtain explicitly the uniform exponential energy decay estimate for the solution of (8.1).

The boundary stabilization for Euler-Bernoulli beam equation has been studied by Chen *et al.* [13], Littman and Markus [55], Krall [39], Chen and Zhou [15], Morgül [66] and Rao [73]. All their investigations have shown the controllability and stabilization of Euler-Bernoulli beam equation, clamped at one end and free at the other, except for feedback damping or control force (viscous damping) on the other end. Littman and Markus [55], and Chen and Zhou [15] in particular, have however shown by calculating the eigenvalues of certain hybrid system that uniform stabilization is not possible because of the inclusion of infinitely large wave number k, during the passage of a wave along the length of the beam. Rao [73] concludes the same by semigroup theory.

The difficulty in proving uniform stability, appears to stem from not imposing any restriction that the beam remains approximately straight during vibration (see Rayleigh [76], Clough and Penzien [17]). Motivated by this consideration, the rate of change in both x and t from the equilibrium position of the displacement $y(x,t)$ remains small, that is to say, $|(\partial^2 y/\partial x \partial t)(x,t)|$ remains small. The implication is that the time rate of variation of small slope remains small and also the gradient of the velocity along the length of the panel remains small. Therefore considering the totality along the length of the panel, we impose the restriction that $\int_0^l(\partial^2 y/\partial x\partial t)^2 dx$ remains small. If we compare this quantity with a similar one, $\int_0^l(\partial^2 y/\partial x^2)^2 dx$ which is actually $2/D$ times the potential energy of bending of the panel and is thus finite, then accordingly the restriction on vibrations satisfying the first equation of (8.1), is assumed to be governed by

$$\int_0^l \left(\frac{\partial^2 y}{\partial x\partial t}\right)^2 dx \leq \frac{D}{ml^2}\int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx \qquad t > t_0, \tag{8.6}$$

for appropriate $y_0(x)$ and $y_1(x)$. Here D/ml^2 is a dimensionality constant. For our purpose, we have assumed (8.6) to hold for time $t > t_0$, where t_0 is finite but may be as large as we please.

8.3 Uniform Boundary Stability Result

Theorem 8.1. Let $y(x,t)$ be a solution of the system (8.1) corresponding to the initial conditions $\{y_0, y_1\}$ for which (8.6) holds and $E(0) < \infty$. Then $E(t)$ satisfies the relation

$$E(t) \leq Me^{-\beta t}E(0), \qquad t \geq 0 \tag{8.7}$$

for some reals $\beta > 0$ and $M \geq 1$.

Proof. Proceeding as in Komornik [90], when $0 \leq t \leq t_0$, where t_0 (may be large enough) is a finite real such that (8.6) holds, we have

$$e^{1-t/t_0} \geq 1.$$

Evidently, we can write from (8.5) that

$$E(t) \leq E(0) \leq e^{1-t/t_0}E(0) = M_1 e^{-\beta_1 t}E(0) \qquad \text{for all} \quad 0 \leq t \leq t_0, \tag{8.8}$$

where $M_1 = e$ and $\beta_1 = 1/t_0$.

For the case $t > t_0$, the proof is as in the following: Let $\epsilon > 0$ be a fixed small constant. We define the scalar-valued function E_ϵ as

$$E_\epsilon(t) = E(t) + \epsilon\rho(t) \tag{8.9}$$

for all $t > t_0$, where

$$\rho(t) = 2m\int_0^l x\frac{\partial y}{\partial t}\frac{\partial y}{\partial x}dx. \tag{8.10}$$

Here the constructed functional $E_\epsilon(t)$ is different from the other forms used in Chapters 6 and 7. In fact, it is to be mentioned that like earlier Chapters, similar type of functional can also be adopted here to obtain the desired result, following by that approach.

Since $(\partial y/\partial x)(0,t) = 0$, by Wirtinger's inequality (cf. Shisha, [80]), we have

$$\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx \le \frac{4l^2}{\pi^2}\int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx, \tag{8.11}$$

and also it can be easily established as (7.24) that

$$\left(\frac{\partial y}{\partial t}(l,t)\right)^2 \le 2\left[\left(\frac{\partial y}{\partial t}(0,t)\right)^2 + l\int_0^l \left(\frac{\partial^2 y}{\partial x \partial t}\right)^2 dx\right]. \tag{8.12}$$

Now from (8.10) we can estimate $\rho(t)$ as

$$\begin{aligned} \left|\rho(t)\right| &\le \frac{4l^2}{\pi}\sqrt{\frac{m}{D}}\int_0^l \left|\sqrt{m}\frac{\partial y}{\partial t}\right|\left|\frac{\pi}{2l}\sqrt{D}\frac{\partial y}{\partial x}\right| dx \\ &\le \frac{2l^2}{\pi}\sqrt{\frac{m}{D}}\int_0^l \left[m\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\pi^2}{4l^2}D\left(\frac{\partial y}{\partial x}\right)^2\right] dx \le \mu_0 E(t) \end{aligned} \tag{8.13}$$

by (8.11) and the energy equation (8.2), where

$$\mu_0 = \frac{4l^2}{\pi}\sqrt{\frac{m}{D}}. \tag{8.14}$$

Thus from (8.9), we find

$$(1-\epsilon\mu_0)E(t) \le E_\epsilon(t) \le (1+\epsilon\mu_0)E(t). \tag{8.15}$$

We now proceed with the differentiation of (8.10) with respect to t and replace $m(\partial^2 y/\partial t^2)$ by $-D(\partial^4 y/\partial x^4)$ and then integrate to obtain

$$\begin{aligned} \frac{d\rho}{dt} &= 2\int_0^l x\left(m\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t \partial x} - D\frac{\partial^4 y}{\partial x^4}\frac{\partial y}{\partial x}\right) dx \\ &= \int_0^l x\frac{\partial}{\partial x}\left[m\left(\frac{\partial y}{\partial t}\right)^2 + D\left(\frac{\partial^2 y}{\partial x^2}\right)^2\right] dx + 2D\int_0^l \frac{\partial y}{\partial x}\frac{\partial^3 y}{\partial x^3} dx \end{aligned} \tag{8.16}$$

Again integrating by parts and applying the boundary conditions of the system (8.1), the above becomes

$$\frac{d\rho}{dt} = m_h\left(\frac{\partial y}{\partial t}(0,t)\right)^2 + ml\left(\frac{\partial y}{\partial t}(l,t)\right)^2 - 2D\int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx - 2E(t). \tag{8.17}$$

Inserting the inequality (8.12) into (8.17), we obtain

$$\frac{d\rho}{dt} \le (m_h + 2ml)\left(\frac{\partial y}{\partial t}(0,t)\right)^2 + 2\left[ml^2\int_0^l \left(\frac{\partial^2 y}{\partial x \partial t}\right)^2 dx - D\int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx\right] - 2E(t) \tag{8.18}$$

and by the use of (8.6), the above ultimately yields

$$\frac{d\rho}{dt} \le (m_h + 2ml)\left(\frac{\partial y}{\partial t}(0,t)\right)^2 - 2E(t). \tag{8.19}$$

Again differentiating (8.9) with respect to t, and inserting (8.4) and (8.19), we obtain the differential inequality

$$\frac{dE_\epsilon}{dt} \leq -2\epsilon E(t) - \left[b - \epsilon(m_h + 2ml)\right]\left(\frac{\partial y}{\partial t}(0,t)\right)^2. \tag{8.20}$$

If we choose $\epsilon \leq \epsilon_0$, where

$$\epsilon_0 = \min\{b/(m_h + 2ml),\ 1/2\mu_0\}, \tag{8.21}$$

then from (8.20), it follows that for all $t > t_0$,

$$\frac{dE_\epsilon}{dt} + 2\epsilon E(t) \leq 0, \tag{8.22}$$

and at the same time we have from (8.15)

$$\mu_0\epsilon E(t) \leq E_\epsilon(t) \leq (1 + \mu_0\epsilon)E(t). \tag{8.23}$$

With the help of (8.23), the relation (8.22) yields

$$\frac{dE_\epsilon}{dt} + \beta_2 E_\epsilon(t) \leq 0, \qquad t > t_0 \tag{8.24}$$

where,

$$\beta_2 = \frac{2\epsilon}{1 + \mu_0\epsilon} > 0. \tag{8.25}$$

Now multiplying (8.24) by $e^{\beta_2 t}$ and integrating over t_0 to t, we obtain

$$E_\epsilon(t) \leq e^{-\beta_2(t-t_0)}E_\epsilon(t_0) \qquad \text{for } t > t_0. \tag{8.26}$$

Then finally, in virtue of (8.23) and (8.5), it follows from (8.26) that

$$E(t) \leq \frac{1 + \mu_0\epsilon}{\mu_0\epsilon}e^{-\beta_2(t-t_0)}E(t_0) \leq M_2 e^{-\beta_2 t}E(0) \qquad \text{for } t > t_0, \tag{8.27}$$

where

$$M_2 = \frac{1 + \mu_0\epsilon}{\mu_0\epsilon}e^{\beta_2 t_0}. \tag{8.28}$$

From the relations (8.8) and (8.27), we can conclude the result (8.7) for some reals $M = \max\{M_1,\ M_2\}$ and $\beta = \min\{\beta_1,\ \beta_2\}$.

8.4 Restrictions on Initial Conditions

The uniform stability for the problem has been established in the previous section on the basis of restriction given by (8.6). From physical point of view, actually, the restriction (8.6) eliminates high wave numbers during vibration, after a time t_0. Therefore, when wave numbers are bounded for the condition (8.6) to hold in terms of initial conditions of the boundary value problem, we use separation of variables method for obtaining the formal result.

The general modal solution of the governing equation of (8.1) by separation of variables is

$$y(x,t) = \sum_{n=0}^{\infty} C_n e^{-ia^2 k_n^2 lt} \psi_n(x, k_n) \tag{8.29}$$

where

$$\psi_n(x,k_n) = \sin k_n x + C_{1n} \cos k_n x + C_{2n} \sinh k_n x + C_{3n} \cosh k_n x \tag{8.30}$$

are the non-orthogonal collection of eigen functions satisfying the boundary value problem (8.1), and a^2 —the 'velocity of propagation'—is $\sqrt{D/ml^2}$. Without loss of generality, we can assume in (8.29) that the wave number k_n ($n = 0, 1, 2, ...$) satisfies $Re\, k_n^2 \geq 0$. The boundary conditions of (8.1) yield the coefficients

$$\begin{aligned} C_{1n} &= \frac{1 + \cos k_n l \cosh k_n l - \sin k_n l \sinh k_n l}{\cos k_n l \sinh k_n l + \sin k_n l \cosh k_n l}, \\ C_{2n} &= -1, \\ C_{3n} &= \frac{1 + \cos k_n l \cosh k_n l + \sin k_n l \sinh k_n l}{\cos k_n l \sinh k_n l + \sin k_n l \cosh k_n l}, \end{aligned} \tag{8.31}$$

and the frequency equation is

$$k_n l(\cos k_n l \sinh k_n l + \sin k_n l \cosh k_n l) + (\alpha a^4 k_n^2 l^3 + i\lambda b a^2 l^2)(1 + \cos k_n l \cosh k_n l) = 0. \tag{8.32}$$

If $k_n^2 = u_n + iv_n$, then we can prove as in Krall [39], that $v_n < 0$. In view of governing differential equation (8.1), separation of variables leads to

$$\psi_n^{iv} - k_n^4 \psi_n = 0 \tag{8.33}$$

where $\psi_n^{iv} = \partial^4 \psi_n / \partial x^4$ and the boundary conditions leading to

$$\begin{aligned} &\psi_n'(0) = 0, && \psi_n'''(0) = (\alpha a^4 k_n^4 l^2 + i\lambda b a^2 k_n^2 l)\psi_n(0), \\ &\psi_n''(l) = 0, && \psi_n'''(l) = 0. \end{aligned} \tag{8.34}$$

Multiplying (8.33) by the complex conjugate $\bar{\psi}_n$ and then taking its conjugate, we find respectively

$$\bar{\psi}_n \psi_n^{iv} - k_n^4 |\psi_n|^2 = 0$$

and

$$\psi_n\bar{\psi}_n^{iv} - \bar{k}_n^4|\psi_n|^2 = 0.$$

Subtracting these two and integrating from 0 to l, we have

$$(k_n^4 - \bar{k}_n^4)\int_0^l |\psi_n|^2 dx = \int_0^l (\bar{\psi}_n\psi_n^{iv} - \psi_n\bar{\psi}_n^{iv})dx.$$

Integrating by parts and applying the boundary conditions in (8.34), we obtain from above after a simplification

$$(k_n^4 - \bar{k}_n^4)\int_0^l |\psi_n|^2 dx = \alpha a^4 l^2(\bar{k}_n^4 - k_n^4)|\psi_n(0)|^2 - i\lambda b a^2 l(k_n^2 + \bar{k}_n^2)|\psi_n(0)|^2. \qquad (8.35)$$

Now if we take $k_n^2 + \bar{k}_n^2 = 2u_n \neq 0$, it follows immediately from (8.35) that

$$v_n = -\frac{1}{2}\frac{\lambda b a^2 l|\psi_n(0)|^2}{\int_0^l |\psi_n|^2 dx + \alpha a^4 l^2|\psi_n(0)|^2} < 0.$$

($v_n \neq 0$, since otherwise $\psi_n(0) = 0$ and then (8.33)-(8.34) yield ψ_n identical to zero). On the other hand, if we take, $k_n^2 + \bar{k}_n^2 = 0$, then $k_n^2 = iv_n$ is purely imaginary. In this case the boundary value problem becomes

$$\begin{aligned}
&\psi_n^{iv} + v_n^2\psi_n = 0 \\
&\psi_n'(0) = 0, \qquad \psi_n'''(0) = -(\alpha a^4 v_n^2 l^2 + \lambda b a^2 v_n l)\psi_n(0), \qquad (8.36)\\
&\psi_n''(l) = 0, \qquad \psi_n'''(l) = 0,
\end{aligned}$$

and since the boundary value problem is real, considering the real solution ψ_n of (8.36) we can write

$$\psi_n\psi_n^{iv} + v_n^2\psi_n^2 = 0.$$

Integrating by parts from 0 to l and applying the boundary conditions in (8.36), we obtain from the above

$$\lambda b a^2 v_n l\psi_n^2(0) = -v_n^2\Big[\int_0^l \psi_n^2 dx + \alpha a^4 l^2\psi_n^2(0)\Big] - \int_0^l (\psi_n'')^2 dx.$$

Thus we have $v_n < 0$ ($v_n \neq 0$, since otherwise the above leads to ψ_n'' identical to zero, which is impossible). Thus the solution (8.29) decays uniformly exponentially as $t \to \infty$, unless $v_n = Im\, k_n^2 \to 0$ as $n \to \infty$.

Restrictions on the initial conditions y_0 and y_1 in terms of wave number k_n, for which inequality (8.6) holds and exponential decay of energy is possible according to the theorem 8.1 are given in the following proposition.

Proposition. Let the associated dimensionless wave numbers $k_n l$ $(n = 0, 1, 2, ...)$ satisfying the frequency equation (8.32) of the modal solution (8.29) and the initial conditions $y_0(x)$ and $y_1(x)$ be such that

$$|k_n^2 l^2|^2 \le \frac{\pi^2}{4} \frac{\sin^2(a^2 u_n lt + \theta_{1n})}{\sin^2(a^2 u_n lt + \theta_{2n})} \qquad \text{for} \quad t > t_0 \tag{8.37}$$

where $t_0 \ge 0$ is a real (which may be large) and θ_{1n}, θ_{2n} are related to the initial conditions by equations (8.41) and (8.44); then the inequality (8.6) holds for $t > t_0$.

Proof. Since the collection of eigen functions ψ_n $(n = 0, 1, 2, ...)$ in (8.30) are not orthogonal, the expansion (8.29) is not an orthogonal expansion. To obtain the coefficients C_n in terms of initial conditions in the simplest way, we need to make the eigen functions orthogonal. Let us consider the real orthonormal eigen functions $\phi_n(x)$ $(n = 0, 1, 2, ...)$ obtained from the modal functions $C_n \psi_n(x, k_n)$ by Gram-Schmidt orthogonalization, so that the modal solution (8.29) can be rewritten as

$$y(x,t) = \sum_{n=0}^{\infty} (A_n + iB_n) e^{-ia^2(u_n + iv_n)lt} \phi_n(x). \tag{8.38}$$

In fact, the solution of the problem (8.1) , is given by

$$Re\, y(x,t) = \sum_{n=0}^{\infty} e^{a^2 v_n lt} \Big[A_n \cos(a^2 u_n lt) + B_n \sin(a^2 u_n lt) \Big] \phi_n(x). \tag{8.39}$$

The constants A_n and B_n are so chosen that at $t = 0$

$$Re\, y(x,0) = y_0 \qquad \text{and} \qquad Re\, y_t(x,0) = y_1. \tag{8.40}$$

For orthonormal expansion of the solution, the coefficients A_n and B_n are given by

$$A_n = \int_0^l y_0 \phi_n dx \qquad \text{and} \qquad u_n B_n + v_n A_n = \frac{1}{a^2 l} \int_0^l y_1 \phi_n dx \tag{8.41}$$

in a simple way. More specifically for the nth term only, we can have from (8.39)

$$\int_0^l \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx = e^{2a^2 v_n lt} \Big[A_n \cos(a^2 u_n lt) + B_n \sin(a^2 u_n lt) \Big]^2 \int_0^l (\phi_n'')^2 dx$$

$$= e^{2a^2 v_n lt} (A_n^2 + B_n^2) \sin^2(a^2 u_n lt + \theta_{1n}) \int_0^l (\phi_n'')^2 dx \tag{8.42}$$

and

$$\int_0^l \left(\frac{\partial^2 y}{\partial x \partial t} \right)^2 dx = e^{2a^2 v_n lt} a^4 l^2 \Big[(u_n B_n + v_n A_n) \cos(a^2 u_n lt) + (v_n B_n - u_n A_n) \sin(a^2 u_n lt) \Big]^2 \int_0^l (\phi_n')^2 dx$$

$$= e^{2a^2 v_n lt} a^4 l^2 (A_n^2 + B_n^2)(u_n^2 + v_n^2) \sin^2(a^2 u_n lt + \theta_{2n}) \int_0^l (\phi_n')^2 dx, \tag{8.43}$$

where

$$\tan\theta_{1n} = \frac{A_n}{B_n} \qquad \text{and} \qquad \tan\theta_{2n} = \frac{u_nB_n + v_nA_n}{v_nB_n - u_nA_n}, \tag{8.44}$$

and $-\pi/2 < \theta_{1n}, \theta_{2n} < \pi/2$. Now by Wirtinger's inequality, we can write

$$\int_0^l (\phi_n')^2 dx \le \frac{4l^2}{\pi^2}\int_0^l (\phi_n'')^2 dx, \tag{8.45}$$

and applying it, (8.43) becomes

$$\int_0^l \left(\frac{\partial^2 y}{\partial x \partial t}\right)^2 dx \le e^{2a^2 v_n lt}\frac{4a^4l^4}{\pi^2}(A_n^2 + B_n^2)(u_n^2 + v_n^2)\sin^2(a^2u_nlt + \theta_{2n})\int_0^l (\phi_n'')^2 dx. \tag{8.46}$$

Suppose there exists a real $t_0 \ge 0$ such that we can impose a restriction on the wave number k_nl determined by (8.32) for $t > t_0$, according to

$$|k_n^2l^2|^2 = (u_n^2 + v_n^2)l^4 \le \frac{\pi^2 \sin^2(a^2u_nlt + \theta_{1n})}{4\sin^2(a^2u_nlt + \theta_{2n})} \tag{8.47}$$

then it follows immediately, from (8.42) and (8.46) that

$$\int_0^l \left(\frac{\partial^2 y}{\partial x \partial t}\right)^2 dx \le a^4 \int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx \qquad \text{for} \quad t > t_0, \tag{8.48}$$

and since $a^2 = \sqrt{D/ml^2}$, the proposition follows.

The condition (8.47) restricts the wave numbers k_nl determined by (8.32). In other words, the initial condition y_0 and y_1 involved in the expressions of θ_{1n} and θ_{2n} are such that after time t_0, the associated wave number $|k_nl|$ falls sufficiently to satisfy (8.47). It should be noted that the right hand side of (8.47) is a positive finite quantity except for values of t for which $a^2u_nlt + \theta_{2n}$ is a multiple of π. For these values of t, it follows from(8.43) that $\int_0^l(\partial^2y/\partial x\partial t)^2dx = 0$ and the inequality (8.48) is valid. Again, for values of t satisfying $a^2u_nlt + \theta_{1n}$ a multiple of π, the expression (8.42) shows that $\int_0^l(\partial^2 y/\partial x^2)^2dx = 0$ and at the same time the right hand side of (8.47) is also zero. Accordingly, because of non-negativity of $|k_n^2l^2|^2$, it follows from (8.47) that $|k_n^2l^2|^2 = (u_n^2 + v_n^2)l^4 = 0$, which implies from (8.43) that $\int_0^l(\partial^2y/\partial x\partial t)^2dx = 0$ for these values of t. Thus the inequality (8.48) is also valid for these values of t.

8.5 Concluding Remarks

It follows from (8.25) that exponential energy decay rate β after passage of the time t_0 will be maximum for largest possible value of ϵ, i.e., for $\epsilon = \epsilon_0$. Choosing ϵ_0 equal to $b/(m_h + 2ml)$ or $1/2\mu_0$ according to (8.21), the maximum energy decay rate β will be equal to either $2b(m_h + 2ml + b\mu_0)^{-1}$ or $2/3\mu_0$, and since as in (8.14),

μ_0 is proportional to l^2, the maximum β decreases quadratically with increasing l after the elapse of the time t_0. Hence it appears, that the decay of the solution of the system will be slower for a longer panel.

Here we have established uniform boundary stabilization of flexural vibrations of a hybrid system consisting of an elastic panel with a movable rigid hub attached at one end, by taking into account a natural restriction for small vibrations (see Rayleigh [76]) of a panel or beam. The uniform boundary stability result of this Chapter, is also valid analogously for an Euler-Bernoulli beam. For arbitrary initial conditions $y_0(x)$ and $y_1(x)$, when the energy and motion decay with time following (8.4) and the panel approaches its straight position, then there comes a time $t_0 > 0$, when the associated wave number $|kl|$ falls sufficiantly to satisfy the relation (8.6) for $t > t_0$. For this reason, we have assumed (8.6) to hold at least in the final stages of vibration after the elapse of the time t_0, however large. For such cases, we have established here the uniform exponential energy decay estimate (8.7) for $t \geq 0$. Our discussion here, has significantly covered the cases of uniform stability of such type of small vibration problems from the mathematical point of view.

CHAPTER 9

BOUNDARY STABILIZATION OF FLEXURAL VIBRATIONS OF AN INTERNALLY DAMPED FLEXIBLE PANEL†

9.1 Introduction and Mathematical Formulation

To study boundary stabilization of the solution of flexural vibrations of the internally damped (Kelvin-Voigt type) flexible rectangular panel as described in Chapter 4, by the system (4.7), we need a boundary stabilizer such that the solution corresponding to initial data with finite energy, decays uniformly exponentially in the energy space as $t \to +\infty$. For this, we now select a viscous damping force on the rigid hub of the panel that means, $Q(t)$ in the system (4.7) is taken as proportional to $(\partial y/\partial t)(0,t)$ say, $Q(t) = b(\partial y/\partial t)(0,t)$ ($b > 0$ is a constant) to describe the asymptotic behavior of the system. Hence for the study of uniform boundary stability for the vibration of internally damped hybrid panel, we are concerned the following system of equations.

$$
\left.
\begin{aligned}
&m\frac{\partial^2 y}{\partial t^2}(x,t) + \mu D\frac{\partial^5 y}{\partial x^4 \partial t}(x,t) + D\frac{\partial^4 y}{\partial x^4}(x,t) = 0, && 0 \le x \le l,\ t \ge 0,\\
&\frac{\partial^3 y}{\partial x^3}(0,t) + \mu\frac{\partial^4 y}{\partial x^3 \partial t}(0,t) + \alpha\frac{\partial^2 y}{\partial t^2}(0,t) + \lambda b\frac{\partial y}{\partial t}(0,t) = 0, && t \ge 0,\\
&\frac{\partial y}{\partial x}(0,t) = 0, \quad \frac{\partial^2 y}{\partial x^2}(l,t) = 0 \quad \frac{\partial^3 y}{\partial x^3}(l,t) = 0, && t \ge 0,\\
&y(x,0) = y_0(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = y_1(x), && 0 \le x \le l,
\end{aligned}
\right\}
\tag{9.1}
$$

where the parameters m, D, μ, α, λ are the same as defined in Chapter 4.

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†The contents of this chapter have been communicated in the form of a paper *Exact Controllability and Boundary Stabilization of Flexural Vibrations of an Internally Damped Flexible Space Structure* —Gorain and Bose, 'Applied Mathematics and Computation'.

9.2 Energy of the System

To each solution of (9.1), the total energy at time t is defined by

$$E(t) = \frac{1}{2}\int_0^l \left[m\left(\frac{\partial y}{\partial t}\right)^2 + D\left(\frac{\partial^2 y}{\partial x^2}\right)^2\right]dx + \frac{1}{2}m_h\left(\frac{\partial y}{\partial t}(0,t)\right)^2. \tag{9.2}$$

Differentiating with respect to t and using the first equation of (9.1), we get

$$E(t) = D\int_0^l \frac{\partial^2 y}{\partial x^2}\frac{\partial^3 y}{\partial x^2 \partial t}dx - D\int_0^l \frac{\partial y}{\partial t}\frac{\partial}{\partial x}\left(\frac{\partial^3 y}{\partial x^3} + \mu\frac{\partial^4 y}{\partial x^3 \partial t}\right)dx + m_h\frac{\partial y}{\partial t}(0,t)\frac{\partial^2 y}{\partial t^2}(0,t).$$

Integrating by parts and applying the boundary conditons in (9.1), we have after a simple calculation

$$\frac{dE}{dt} = -\mu\int_0^l D\left(\frac{\partial^3 y}{\partial x^2 \partial t}\right)^2 dx - b\left(\frac{\partial y}{\partial t}(0,t)\right)^2, \tag{9.3}$$

since $\lambda = 1/D$. The negativity of the right hand side of (9.3) shows that the energy $E(t)$ of the system (9.1) is nonincreasing with time and the system is a non-conserving system. The integral term in (9.3), ensures that the energy of the system (9.1) decays due to incorporation of Kelvin-Voigt type material damping of the structure, while the last term is due to viscous boundary damping at the hub end. Thus we have

$$E(t) \leq E(0) \qquad \text{for } t \geq 0. \tag{9.4}$$

9.3 Uniform Boundary Stability Result

There are several authors (cf. Chen *et al.* [13], Chen and Zhou [15], Littman and Markus [55], Morgül [66] and Rao [73], to name but a few) who have studied the boundary stability of the vibrations of various types of flexible space structures governed by undamped Euler-Bernoulli beam equation with a clamped end. They have discussed the limitations of the asymptotic behavior of the solution of the system together with the uniform stability. However in the previous Chapter, we have ratified the uniform boundary stability for this type of specific hybrid problem, manifested by a constraint of small vibrations. The objective in this context is to substantiate explicitly the uniform exponential energy decay rate of vibrations governed by the system (9.1), that means we want to set up the result of the form

$$E(t) \leq Me^{-\beta t}E(0) \qquad t \geq 0, \tag{9.5}$$

for some reals $\beta > 0$ and $M \geq 1$.

The uniform exponential energy decay for the solution of (9.1) follows from the ensuing Theorem:

Theorem 9.1. Let $y(x,t)$ be a solution of (9.1) corresponding to the initial conditions $\{y_0, y_1\}$ with finite energy, i.e., $E(0) < \infty$. Then the energy of the system (9.1) defined by (9.2) decays exponentially with time, i.e., $E(t)$ satisfies (9.5) for some positive β, depending explicitly on the material damping parameter μ, and some real $M \geq 1$.

Proof. Let $\epsilon > 0$ be small but fixed constant. We define a function $E_\epsilon(t)$ by

$$E_\epsilon(t) = E(t) + \epsilon\mu\rho(t), \qquad t \geq 0, \tag{9.6}$$

where

$$\rho(t) = m\int_0^l y\frac{\partial y}{\partial t}dx + \frac{1}{2}\mu\int_0^l D\Big(\frac{\partial^2 y}{\partial x^2}\Big)^2 dx + m_h y(0,t)\frac{\partial y}{\partial t}(0,t). \tag{9.7}$$

Now differentiating (9.7) with respect to t and replacing $m(\partial^2 y/\partial t^2)$ by $-\mu D(\partial^5 y/\partial x^4 \partial t) - D(\partial^4 y/\partial x^4)$, we get

$$\begin{aligned}\frac{d\rho}{dt} &= m\int_0^l \Big(\frac{\partial y}{\partial t}\Big)^2 dx - D\int_0^l y\frac{\partial}{\partial x}\Big(\frac{\partial^3 y}{\partial x^3} + \mu\frac{\partial^4 y}{\partial x^3 \partial t}\Big)dx \\ &+\mu\int_0^l D\frac{\partial^2 y}{\partial x^2}\frac{\partial^3 y}{\partial x^2 \partial t}dx + m_h\Big(\frac{\partial y}{\partial t}(0,t)\Big)^2 + m_h y(0,t)\frac{\partial^2 y}{\partial t^2}(0,t).\end{aligned}$$

Integrating by parts, we have from above

$$\begin{aligned}\frac{d\rho}{dt} &= m\int_0^l \Big(\frac{\partial y}{\partial t}\Big)^2 dx - D\Big[y\Big(\frac{\partial^3 y}{\partial x^3} + \mu\frac{\partial^4 y}{\partial x^3 \partial t}\Big) - \frac{\partial y}{\partial x}\Big(\frac{\partial^2 y}{\partial x^2} + \mu\frac{\partial^3 y}{\partial x^2 \partial t}\Big)\Big]_0^l \\ &-D\int_0^l \Big(\frac{\partial^2 y}{\partial x^2}\Big)^2 dx + m_h\Big(\frac{\partial y}{\partial t}(0,t)\Big)^2 + m_h y(0,t)\frac{\partial^2 y}{\partial t^2}(0,t).\end{aligned}$$

Applying the boundary conditions (9.1), it yields

$$\frac{d\rho}{dt} = m\int_0^l \Big(\frac{\partial y}{\partial t}\Big)^2 dx - \int_0^l D\Big(\frac{\partial^2 y}{\partial x^2}\Big)^2 dx + m_h\Big(\frac{\partial y}{\partial t}(0,t)\Big)^2, \tag{9.8}$$

since $\alpha = m_h/D$, $\lambda = 1/D$. We note that

$$\Big|m\int_0^l y\frac{\partial y}{\partial t}dx\Big| \leq \frac{m}{2}\int_0^l \Big[\Big(\frac{\partial y}{\partial t}\Big)^2 + y^2\Big]dx \tag{9.9}$$

and

$$\Big|m_h y(0,t)\frac{\partial y}{\partial t}(0,t)\Big| \leq \frac{m}{2}\Big[\Big(\frac{\partial y}{\partial t}(0,t)\Big)^2 + \big(y(0,t)\big)^2\Big]. \tag{9.10}$$

Again by Wirtinger inequality (cf. [80]), we have

$$\int_0^l \big[y(x,t) - y(0,t)\big]^2 dx \leq \frac{4l^2}{\pi^2}\int_0^l \Big(\frac{\partial y}{\partial x}(x,t)\Big)^2 dx \tag{9.11}$$

which can be written as

$$\int_0^l \big(y(x,t)\big)^2 dx + l\big(y(0,t)\big)^2 \le 2y(0,t)\int_0^l y(x,t)dx + \frac{4l^2}{\pi^2}\int_0^l \Big(\frac{\partial y}{\partial x}(x,t)\Big)^2 dx$$

$$\le 2l\big(y(0,t)\big)^2 + \frac{1}{2l}\Big(\int_0^l y(x,t)dx\Big)^2 + \frac{4l^2}{\pi^2}\int_0^l \Big(\frac{\partial y}{\partial x}(x,t)\Big)^2 dx. \tag{9.12}$$

Applying the inequality of the form

$$\Big(\int_0^l fg dx\Big)^2 \le \int_0^l f^2 dx \int_0^l g^2 dx, \tag{9.13}$$

we have from (9.12)

$$\int_0^l y^2 dx \le 2l\Big[\big(y(0,t)\big)^2 + \frac{4l}{\pi^2}\int_0^l \Big(\frac{\partial y}{\partial x}\Big)^2 dx\Big]. \tag{9.14}$$

Also since $(\partial y/\partial x)(0,t) = 0$, we have

$$\int_0^l \Big(\frac{\partial y}{\partial x}\Big)^2 dx \le \frac{4l^2}{\pi^2}\int_0^l \Big(\frac{\partial^2 y}{\partial x^2}\Big)^2 dx. \tag{9.15}$$

and there exists a constant $K > 1$, independent of t such that

$$\big(y(0,t)\big)^2 \le K l^3 \int_0^l \Big(\frac{\partial^2 y}{\partial x^2}\Big)^2 dx. \tag{9.16}$$

By the use of (9.9)–(9.10), (9.14)–(9.16), we can write

$$\Big|m\int_0^l y\frac{\partial y}{\partial t}dx + m_h y(0,t)\frac{\partial y}{\partial t}(0,t)\Big| \le R_0 E(t), \tag{9.17}$$

for some suitable positive constant R_0, independent of both t and μ, and

$$0 \le \frac{1}{2}\mu\int_0^l D\Big(\frac{\partial^2 y}{\partial x^2}\Big)^2 dx \le \mu E(t) \tag{9.18}$$

in virtue of (9.2). Thus from (9.6), we find

$$(1-\epsilon\mu R_0)E(t) \le E_\epsilon(t) \le \big[1+\epsilon\mu(R_0+\mu)\big]E(t) \tag{9.19}$$

with the help of the relations (9.7), (9.17) and (9.18). Again differentiating (9.6) with respect to t and using (9.3) and (9.8), we obtain

$$\begin{aligned}
\frac{dE_\epsilon}{dt} &= \mu\Big[\epsilon\int_0^l m\Big(\frac{\partial y}{\partial t}\Big)^2 dx - \int_0^l D\Big(\frac{\partial^3 y}{\partial x^2\partial t}\Big)^2 dx\Big] \\
&\quad -\epsilon\mu\int_0^l D\Big(\frac{\partial^2 y}{\partial x^2}\Big)^2 dx + (\epsilon\mu m_h - b)\Big(\frac{\partial y}{\partial t}(0,t)\Big)^2 \\
&\le \mu\Big[\frac{3\epsilon}{2}\int_0^l m\Big(\frac{\partial y}{\partial t}\Big)^2 dx - \int_0^l D\Big(\frac{\partial^3 y}{\partial x^2\partial t}\Big)^2 dx\Big] + \Big(\frac{3\epsilon\mu m_h}{2} - b\Big)\Big(\frac{\partial y}{\partial t}(0,t)\Big)^2 \\
&\quad -\frac{\epsilon\mu}{2}\Big[\int_0^l \Big(m\Big(\frac{\partial y}{\partial t}\Big)^2 + D\Big(\frac{\partial^2 y}{\partial x^2}\Big)^2\Big)dx + m_h\Big(\frac{\partial y}{\partial t}(0,t)\Big)^2\Big].
\end{aligned} \tag{9.20}$$

In the same way in virtue of (9.14) and (9.15), we can have

$$\int_0^l \Big(\frac{\partial y}{\partial t}\Big)^2 dx \le 2l\Big[\Big(\frac{\partial y}{\partial t}(0,t)\Big)^2 + \frac{16l^3}{\pi^4}\int_0^l \Big(\frac{\partial^3 y}{\partial x^2 \partial t}\Big)^2 dx\Big]. \tag{9.21}$$

Utilizing (9.21), the relation (9.20) can be written as

$$\begin{aligned}\frac{dE_\epsilon}{dt} &\le \mu\Big[48\epsilon m l^4/\pi^4 - D\Big]\int_0^l \Big(\frac{\partial^3 y}{\partial x^2 \partial t}\Big)^2 dx \\ &\quad + \Big[3\epsilon\mu(ml + m_h/2) - b\Big]\Big(\frac{\partial y}{\partial t}(0,t)\Big)^2 - \epsilon\mu E(t).\end{aligned} \tag{9.22}$$

If we choose $\epsilon \le \epsilon_0$, where

$$\epsilon_0 = \min\Big\{D\pi^4/48ml^4,\ 2b/(6\mu ml + 3\mu m_h),\ 1/(\mu R_0 + \mu^2)\Big\}, \tag{9.23}$$

then from (9.22), it follows that

$$\frac{dE_\epsilon}{dt} + \epsilon\mu E(t) \le 0 \tag{9.24}$$

and at the same time we have from (9.19)

$$\mu^2\epsilon E(t) \le E_\epsilon(t) \le \mu_0 E(t), \tag{9.25}$$

where

$$\mu_0 = 1 + \epsilon R_0 \mu + \epsilon\mu^2 \tag{9.26}$$

is a quadratic function of μ. Introducing (9.25) into (9.24), yields

$$\frac{dE_\epsilon}{dt} + \beta E_\epsilon(t) \le 0, \tag{9.27}$$

where

$$\beta = \frac{\epsilon\mu}{\mu_0} > 0. \tag{9.28}$$

Now multiplying (9.28) by $e^{\beta t}$ and integrating over 0 to t, we obtain

$$E_\epsilon(t) \le e^{-\beta t}E_\epsilon(0), \qquad \text{for } t \ge 0. \tag{9.29}$$

Then finally, with the help of (9.25), it follows from (9.29) that

$$E(t) \le Me^{-\beta t}E(0) \qquad \text{for } t \ge 0, \tag{9.30}$$

where

$$M = \frac{\mu_0}{\mu^2\epsilon} \ge 1. \tag{9.31}$$

9.4 Concluding Remarks

Since the energy $E(t)$ defined by (9.2) of the system (9.1) decays exponentially, i.e., $E(t) \to 0$ as $t \to +\infty$, we conclude that the solution of the system decays uniformly exponentially with time. It is obvious that the energy decay rate will be maximum for largest admissible value of β. From the relations (9.26) and (9.28) as we have $\beta = \epsilon\mu/(1+\epsilon R_0\mu+\epsilon\mu^2)$, it follows that β increases with the increasing values of ϵ. The expression for β as a function of viscoelastic damping parameter μ shows that the decay rate β will be maximum for $\mu = 1/\sqrt{\epsilon}$. For this value of μ, we thus have $\beta_{\max} = \sqrt{\epsilon}/(2+R_0\sqrt{\epsilon})$, which will be larger for greater admissible value of ϵ. As ϵ is bounded above by ϵ_0 given by (9.23), eventually we have $\beta_{\max} = \sqrt{\epsilon_0}/(2+R_0\sqrt{\epsilon_0})$.

In this Chapter, we have studied the uniform boundary stability of flexural vibrations of an internally damped flexible panel hoisted by a rigid hub at one end and totally free at the other end. The hub dynamics leads to a non-standard boundary condition and the overall system becomes a hybrid model. To make the problem more realistic, internal material damping of Voigt-type has been incorporated. It has been shown that, by applying a boundary stabilizer (viscous damping) at the hub end only, uniform stabilization of the vibrations of the system is possible without applying any constraints at the free end. By analogy, the method is applicable to an Euler-Bernoulli beam instead of the panel and in this case the flexural rigidity D of the panel will be replaced by EI of the beam.

CHAPTER 10

UNIFORM STABILITY OF INTERNALLY DAMPED WAVE EQUATION IN A BOUNDED DOMAIN IN $\mathbf{R}^n$ †

10.1 Introduction

In the previous Chapters, we have particularly considered one dimensional torsional or flexural vibrations of hybrid models of dynamics. We have discussed the uniform boundary stability for the solution of such systems with the demonstration of both internally damped and internally undamped flexible structures as models. We have observed the relevence of the boundary stabilizer at one end with the other end totally free, to procure the asymptotic behavior of the solutions of the systems. In this Chapter, we are concerned about the uniform stability of the solution of internally damped wave equation $y'' = \Delta y + \mu \Delta y'$ with small damping constant $\mu > 0$, in a bounded domain Ω in $\mathbf{R}^n$ under mixed undamped boundary conditions. Here the prime denotes the differentiation with respect to time and Δ the Laplacian in $\mathbf{R}^n$. This mathematical equation is simple but realistic for the dynamics of flexible mechanical structure and is known as 'Kelvin-Voigt' model of viscoelasticity in which a linear spring is connected with a dashpot in series (cf. Fung [25], Rabotnov [71]). In this case, the differential relation between the stress σ and strain e is given by $\sigma = E(e + \mu e')$ which is better than ordinary Hook's law for real elastic structures, E being the modulus of elasticity of the structure. Earlier investigations have considered the undamped wave equation with certain forms of damped boundary conditions proving similar and faster energy decay rates.

†The contents of this chapter has been published in the paper *Exponential Energy Decay Estimate for the Solutions of Internally Damped Wave Equation in a Bounded Domain* —Gorain, 'Journal of Mathematical Analysis and Applications', Vol. 216, 510–520, (1997).

10.2 Mathematical Formulation

Let Ω be a bounded connected set in $\mathbf{R}^n$ and let Γ be its boundary which is piecewise smooth consisting of two parts Γ_0 and Γ_1 such that $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$ and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$. We denote by ν, the unit normal of Γ pointing towards exterior of Ω. Let x^0 be an arbitary but fixed point in $\mathbf{R}^n$ and set

$$m(x) = x - x^0, \qquad x \in \mathbf{R}^n. \tag{10.1}$$

Let the two disjoint open subsets Γ_1 and Γ_0 of Γ be defined as

$$m(x).\nu(x) > 0 \qquad \text{on} \quad \Gamma_1 \tag{10.2}$$

$$m(x).\nu(x) \leq 0 \qquad \text{on} \quad \Gamma_0, \tag{10.3}$$

where . denotes the scalar product in $\mathbf{R}^n$.

Herein we shall study the uniform stability of the internally damped wave equation

$$y'' = \Delta y + \mu \Delta y' \qquad \text{in} \quad \Omega \times (0, \infty) \tag{10.4}$$

with the following undamped mixed boundary conditions

$$y = 0 \qquad \text{on} \quad \Gamma_0 \times (0, \infty) \tag{10.5}$$

$$\partial y / \partial \nu = 0 \qquad \text{on} \quad \Gamma_1 \times (0, \infty) \tag{10.6}$$

and initial conditions

$$y(0) = y_0 \;\text{ and }\; y'(0) = y_1 \qquad \text{in} \quad \Omega \tag{10.7}$$

where $'$ denote the time derivative, Δ the Laplacian in $\mathbf{R}^n$ taken in the space variables and $\mu > 0$ is the small internal damping constant.

Physically, equation (10.4) occurs in the study of vibrations of flexible structures in a bounded domain governed by the Kelvin-Voigt model of viscoelasticity. The motivation for incorporating internal material damping in the wave equation as in (10.4) arises from the fact that, inherent small material damping, usually uniform of constant measure (μ in the Voigt model), is always present in real materials (cf. Christensen [16]). Hence from the physical point of view we say that internal structural damping force will appear so long as the system vibrates.

10.3 Energy of the System

The total energy $E(t)$ for the solution of the system (10.4)–(10.7) defind by

$$E(t) = \frac{1}{2}\int_{\Omega}(y'^2 + |\nabla y|^2)dx. \qquad (10.8)$$

The question of uniform exponential decay of energy of the solution of the undamped wave equation in Ω has been studied by a number of authors—Chen [6], Lagnese [42], Lasiecka and Triggiani [46], Triggiani [86], and Lions [52]. They considerd the following prototype system :

$$y'' = \Delta y \qquad \text{in} \quad \Omega \times (0, \infty) \qquad (10.9)$$

$$y = 0 \qquad \text{on} \quad \Gamma_0 \times (0, \infty) \qquad (10.10)$$

$$\partial y/\partial \nu = -b(x)y' \qquad \text{on} \quad \Gamma_1 \times (0, \infty) \qquad (10.11)$$

$$y(0) = y_0 \quad \text{and} \quad y'(0) = y_1 \qquad \text{in} \quad \Omega \qquad (10.12)$$

where $b(x) \in L^{\infty}(\Gamma_1)$, $b(x) \geq b_0 > 0$; that is boundary damping is essential on some portion $\Gamma_1(\Gamma_1 \neq \phi)$ of the boundary Γ. They proved result of the form

$$E(t) \leq Me^{-\beta t}E(0), \qquad t \geq 0 \qquad (10.13)$$

$M \geq 1$ and $\beta > 0$ being some constants. Later, Lagnese [43] and Komornik [37] obtained somewhat faster energy decay rates for certain forms of $b(x)$. Also Chen [9] demonstrated faster energy decay rate when external damping $2\gamma y'$ is present in the left hand side of (10.9). In the method of treatment [9,37,43] adopt direct method by constructing suitable functionals related to $E(t)$, where as [6,42,86] employ semigroup theory (cf. Pazy [69]) in as much as the underlying operator of the system generates strongly continuous contraction semigroup.

Establishment of uniform exponential energy decay of the form (10.13) is now sought under natural boundary conditions (10.5) and (10.6), without having to introduce boundary damping. Here, the exponential decay rate will depend on μ and we adopt a direct method such as in Komornik [37], Lagnese [43] for extracting the functional form of this dependence. In contrast, Chen and Russell [14] considerd generalised operator version of (10.4) of the form $y'' + \mathbf{B}y' + \mathbf{A}y = 0$ to study the analyticity of the semigroup of contraction over suitable Hilbert space of the underlying operator. Also, several examples of partial differential equations with boundary or point control have been illustrated in Lasiecka and Triggiani [49] which can be reduced to the abstract form.

Now by differentiating equation (10.8) with respect to t and replacing y'' by $\Delta y + \mu\Delta y'$ from the governing equation (10.4), we obtain

$$E'(t) = \int_\Omega \left[y'(\Delta y + \mu\Delta y') + (\nabla y.\nabla y')\right] dx.$$

Applying Green's formula we have

$$\begin{aligned} E'(t) &= \int_\Gamma y'\left(\frac{\partial y}{\partial\nu} + \mu\frac{\partial y'}{\partial\nu}\right)d\Gamma - \mu\int_\Omega |\nabla y'|^2 dx \\ &= -\mu\int_\Omega |\nabla y'|^2 dx, \end{aligned} \tag{10.14}$$

where the boundary conditions (10.5)–(10.6) have been used. We thus have

$$E'(t) \leq 0 \qquad \text{for} \quad t \geq 0. \tag{10.15}$$

Hence energy is nonincreasing with time, i.e.,

$$E(t) \leq E(0) \qquad \text{for} \quad t \geq 0. \tag{10.16}$$

We establish from the negativity of the right hand side of (10.14) that the energy of the system is dissipating due to the presence of internal material damping of the system.

10.4 Uniform Stability Result

The validity of the uniform exponential decay of $E(t)$ for the problem (10.4)–(10.7) follows from the Theorem:

Theorem 10.1. Let y be a regular solution of (10.4)–(10.7). Then the energy

$$E(t) \leq Me^{-\beta t}E(0), \qquad t \geq 0$$

for some reals $M \geq 1$, $\beta > 0$ of the form $\beta = \mu/(a\mu^2 + b\mu + c)$, $a, b, c > 0$ and for all initial states $y_0 \in H^2_{\Gamma_0}(\Omega)$, $y_1 \in H^1(\Omega)$ where, $H^2_{\Gamma_0}(\Omega) = \{y \mid y \in H^2(\Omega),\ y = 0 \text{ on } \Gamma_0\}$ and $H^k(\Omega)$, k being positive integer, is the classical Sobolev space of real valued functions y whose partial derivatives defined in the distributional sense of order $\leq k$ lie in $L^2(\Omega)$.

Before proving the above main Theorem, we first establish the followings.

Lemma 10.1. If y be a regular solution of the equations (10.4)–(10.7), then the funtion $\rho_1(t)$ defind by

$$\rho_1(t) = \frac{1}{2}\int_\Omega \left(|\nabla y'|^2 + |\Delta y|^2\right)dx \tag{10.17}$$

is nonincreasing for $t \geq 0$.

Proof. Differentiating (10.17) with respect to t we obtain

$$\rho'_1(t) = \int_\Omega \left[(\nabla y' . \nabla y'') + \Delta y \Delta y'\right] dx.$$

Use of (10.4) the above yields

$$\rho'_1(t) = \int_\Gamma y'' \frac{\partial y'}{\partial \nu} d\Gamma - \mu \int_\Omega |\Delta y'|^2 dx \tag{10.18}$$

where we have used the Green's formula. Further, using the boundary conditions (10.5) and (10.6) we have

$$\rho'_1(t) = -\mu \int_\Omega |\Delta y'|^2 dx \leq 0. \tag{10.19}$$

Hence $\rho_1(t)$ is nonincreasing for $t \geq 0$. We conclude that

$$\rho_1(t) \leq \rho_1(0) \qquad \text{for } t \geq 0. \tag{10.20}$$

Lemma 10.2. Let y be a regular solution of (10.4)–(10.7). If we define a function $F(t)$ by

$$F(t) = \int_\Omega \left[\nabla y' . (m.\nabla)\nabla y\right] dx \tag{10.21}$$

then $|F(t)| \leq K\rho_1(t)$ for $t \geq 0$, where $K \geq 1$ is a constant, independent of t.

Proof. From equation (10.21) we can write

$$|F(t)| \leq \int_\Omega |\nabla y'||(m.\nabla)\nabla y| dx \leq \frac{1}{2}\int_\Omega \left(|\nabla y'|^2 + |(m.\nabla)\nabla y|^2\right) dx. \tag{10.22}$$

We now define a constant $K \geq 1$ so that we can write

$$\int_\Omega \left|(m.\nabla)\nabla y\right|^2 dx \leq K \int_\Omega |\Delta y|^2 dx. \tag{10.23}$$

It follows from (10.22) then

$$|F(t)| \leq K\rho_1(t) \qquad \text{for } t \geq 0. \tag{10.24}$$

Hence the Lemma follows.

The inequality of the form (10.23) can be written due to the fact that the expressions $m_k(\partial^2 y/\partial x_k \partial x_j)$ can be reduced to the form $M_k(\partial^2 y/\partial x_k^2)$ by suitable orthogonal trasformation of the axes x_j (j,k=1,2,...,n) and Δy is invarient with respect to orthogonal transformation (cf. Edwards [20]), the usual summation convention of repeated indices being used.

Lemma 10.3. For every $y \in H^1(\Omega)$

$$\int_\Omega \left[2y.(m.\nabla)y + n|y|^2\right]dx = \int_\Gamma m.\nu|y|^2 d\Gamma. \tag{10.25}$$

Proof. We have

$$\begin{aligned}\int_\Omega \left[2y.(m.\nabla)y + n|y|^2\right]dx &= \int_\Omega \left[(m.\nabla|y|^2) + n|y|^2\right]dx \\ &= \int_\Omega \text{div}(m|y|^2)dx \\ &= \int_\Gamma m.\nu|y|^2 d\Gamma.\end{aligned}$$

Hence the Lemma.

Lemma 10.4. If y be a regular solution of (10.4)–(10.7), then

$$\begin{aligned}&\rho'(t) + \mu\rho_0'(t) + 2\mu F(t) + 2E(t) \\ &\leq \mu^2 \int_{\Gamma_0} |m.\nu||\nabla y'|^2 d\Gamma + \int_{\Gamma_1} m.\nu y'^2 d\Gamma\end{aligned} \tag{10.26}$$

where,

$$\rho(t) = \int_\Omega \left[2y'(m.\nabla y) + (n-1)yy'\right]dx \tag{10.27}$$

and

$$\rho_0(t) = \frac{n+1}{2}\int_\Omega |\nabla y|^2 dx. \tag{10.28}$$

Proof. Differentiating (10.27) with respect to t and replacing y'' by $\Delta y + \mu\Delta y'$ we have

$$\rho'(t) = \int_\Omega \left[2(\Delta y + \mu\Delta y')(m.\nabla y) + (n-1)(\Delta y + \mu\Delta y')y + 2y'(m.\nabla y') + (n-1)y'^2\right]dx.$$

Applying Green's formula we obtain

$$\begin{aligned}\rho'(t) &= \int_\Gamma \left[(2(m.\nabla y) + (n-1)y\right]\left[\frac{\partial y}{\partial\nu} + \mu\frac{\partial y'}{\partial\nu}\right]d\Gamma \\ &\quad - \int_\Omega \left[2\nabla(m.\nabla y) + (n-1)\nabla y\right].\nabla(y + \mu y')dx \\ &\quad + \int_\Omega \left[2y'(m.\nabla y') + (n-1)y'^2\right]dx.\end{aligned}$$

Using the boundary conditions (10.5) and (10.6) we have

$$\begin{aligned}\rho'(t) &= \int_{\Gamma_0} 2\Big(\frac{\partial y}{\partial \nu} + \mu \frac{\partial y'}{\partial \nu}\Big)(m.\nabla y)d\Gamma \\ &\quad - \int_\Omega \Big[2\big(\nabla y + (m.\nabla)\nabla y\big) + (n-1)\nabla y\Big].\nabla(y + \mu y')dx \\ &\quad + \int_\Omega \Big[2y'(m.\nabla y') + (n-1)y'^2\Big]dx \\ &= \int_{\Gamma_0} 2\Big(\frac{\partial y}{\partial \nu} + \mu \frac{\partial y'}{\partial \nu}\Big)(m.\nabla y)d\Gamma - \int_\Omega \Big[2(\nabla y.(m.\nabla)\nabla y) + n|\nabla y|^2\Big]dx \\ &\quad + \int_\Omega \Big[2y'(m.\nabla)y' + ny'^2\Big]dx - 2\mu \int_\Omega \Big[\nabla y'.(m.\nabla)\nabla y\Big]dx \\ &\quad -\mu(n+1)\int_\Omega (\nabla y.\nabla y')dx - \int_\Omega \Big(|\nabla y|^2 + y'^2\Big)dx,\end{aligned}$$

since $\nabla(m.\nabla y) = \nabla y + (m.\nabla)\nabla y$. Applying Lemma 10.3 for ∇y and y', we obtain from above

$$\rho'(t) = \int_{\Gamma_0} 2\Big(\frac{\partial y}{\partial \nu} + \mu \frac{\partial y'}{\partial \nu}\Big)(m.\nabla y)d\Gamma + \int_\Gamma m.\nu\Big(y'^2 - |\nabla y|^2\Big)d\Gamma$$
$$- 2\mu F(t) - \mu\rho_0{}'(t) - 2E(t), \tag{10.29}$$

with the help of (10.8), (10.21), (10.28). Since $y = 0$ on Γ_0, $\nabla y = \nu(\partial y/\partial \nu)$ and $|\nabla y|^2 = |(\partial y/\partial \nu)|^2$ on Γ_0. Also $m.\nu > 0$ on Γ_1. Hence we have from (10.29)

$$\begin{aligned}\rho'(t) &+ \mu\rho'_0(t) + 2\mu F(t) + 2E(t) \\ &\le \int_{\Gamma_0} m.\nu|\nabla y|^2 d\Gamma + 2\mu\int_{\Gamma_0} m.\nu(\nabla y.\nabla y')d\Gamma + \int_{\Gamma_1} m.\nu y'^2 d\Gamma \\ &\le \int_{\Gamma_0} m.\nu|\nabla y|^2 d\Gamma + \int_{\Gamma_0} |m.\nu|\Big(|\nabla y|^2 + \mu^2|\nabla y'|^2\Big)d\Gamma + \int_{\Gamma_1} m.\nu y'^2 d\Gamma \\ &\le \mu^2\int_{\Gamma_0} |m.\nu||\nabla y'|^2 d\Gamma + \int_{\Gamma_1} m.\nu y'^2 d\Gamma\end{aligned}$$

since, $m.\nu \le 0$ on Γ_0, $m.\nu + |m.\nu| = 0$ on Γ_0. Hence the Lemma.

We are now ready to prove the main result.

Proof of the Theorem. We define a function $G(t)$ for all $t \ge 0$ as

$$G(t) = \lambda E(t) + \mu\Big[\rho(t) + \mu\rho_0(t) + \rho_1(t)\Big] \tag{10.30}$$

where λ is a positive constant defined by

$$\int_{\Gamma_1} m.\nu y'^2 d\Gamma \le \lambda \int_\Omega |\nabla y'|^2 dx \tag{10.31}$$

for all $y' \in H^1_{\Gamma_0}(\Omega)$. We also define the positive constants λ_0, λ_1, and λ_2 by

$$E(t) \le \lambda_0 \rho_1(t) \tag{10.32}$$

$$\int_\Omega y^2 dx \le \lambda_1 \int_\Omega |\nabla y|^2 dx \qquad (\lambda_1 > 1) \tag{10.33}$$

and

$$\int_{\Gamma_0} |m.\nu||\nabla y'|^2 d\Gamma \le \lambda_2 \int_\Omega |\Delta y'|^2 dx \tag{10.34}$$

for all $y \in H^2_{\Gamma_0}(\Omega)$. Inequalities (10.32) and (10.33) arise due to Poincare. Inequalities (10.31) and (10.34) follow from the combination of Poincare inequality with the Trace inequality in $H^2(\Omega)$ (cf. Aubin [1]). Here all λ, λ_0, λ_1 and λ_2 are independent of t, depending only on the set Ω in $\mathbf{R}^n$ and eventually on x^0. They are also independent of initial value of $\{y_0, y_1\}$. Their explicit determination is in general very difficult.

Now we have from (10.27)

$$\begin{aligned} |\rho(t)| &\le R_0 \int_\Omega \left(y'^2 + |\nabla y|^2\right) dx + \frac{(n-1)}{2} \int_\Omega \left(y^2 + y'^2\right) dx \\ &\le \left[2R_0 + (n-1)\lambda_1\right] E(t) = C_0 E(t) \end{aligned} \tag{10.35}$$

where, $R_0 = \sup\{|m(x)| \ : \ x \in \Omega\}$ and $C_0 = [2R_0 + (n-1)\lambda_1]$. From (10.28) we also have

$$0 \le \rho_0(t) \le (n+1)E(t). \tag{10.36}$$

With the help of (10.32), (10.35) and (10.36), it follows from (10.30) that

$$\begin{aligned} (\lambda + \mu/\lambda_0 - \mu C_0)E(t) &\le G(t) \\ &\le \left[\lambda + \mu(C_0 + \mu(n+1))\right] E(t) + \mu\rho_1(t). \end{aligned} \tag{10.37}$$

Now differentiating (10.30) with respect to t and applying (10.14), (10.19) and Lemma 10.4, we have

$$\begin{aligned} G'(t) &= \lambda E'(t) + \mu\left[(\rho'(t) + \mu\rho'_0(t) + \rho'_1(t)\right] \\ &\le -\lambda\mu \int_\Omega |\nabla y'|^2 dx + \mu\Big[\mu^2 \int_{\Gamma_0} |m.\nu||\nabla y'|^2 d\Gamma + \int_{\Gamma_1} m.\nu y'^2 d\Gamma \\ &\quad -2\mu F(t) - 2E(t) - \mu \int_\Omega |\Delta y'|^2 dx\Big]. \end{aligned}$$

Applying the inequalities (10.31), (10.34) and the Lemma 10.2, we obtain

$$G'(t) \le \mu^2(\mu\lambda_2 - 1) \int_\Omega |\Delta y'|^2 dx + 2\mu\left[\mu K \rho_1(t) - E(t)\right]. \tag{10.38}$$

Now since $\{y_0, y_1\} \in H^2_{\Gamma_0}(\Omega) \times H^1(\Omega)$, therefore, from the inequalities (10.20) and (10.16), we have $\rho_1(t) \le \rho_1(0) < \infty$ and $E(t) \le E(0) < \infty$. Hence there exists a positive constant K_0 such that

$$\rho_1(t) \le K_0 E(t). \tag{10.39}$$

We can assert that K_0 is finite, provided that $\sup\{\rho_1(t)/E(t)\}$ is finite for all $t \ge 0$. We can define a finite constant $K^* > 1$ by the Poincare inequality

$$\int_\Omega |\nabla y'|^2 dx \le K^* \int_\Omega |\Delta y'|^2 dx \tag{10.40}$$

for all $y' \in H^2_{\Gamma_0}(\Omega)$. Applying this inequality, we have from (10.14) and (10.19), $E'(t) \ge K^* {\rho'}_1(t)$. As $E'(t)$ and ${\rho'}_1(t)$ are both negative, we conclude that $\rho_1(t)$ decreases faster than $E(t)$ and the assertion follows.

By (10.39), the inequality (10.38) can then be written as

$$G'(t) \le \mu^2(\mu\lambda_2 - 1)\int_\Omega |\Delta y'|^2 dx + \mu(2\mu K K_0 - 1)E(t) - \mu E(t). \tag{10.41}$$

Let

$$\mu \le \min\{1/\lambda_2,\ 1/2KK_0,\ \lambda/C_0\}, \tag{10.42}$$

which determines here an upper bound of the value of μ consistent with stability, we then have from (10.41)

$$G'(t) \le -\mu E(t) \tag{10.43}$$

and at the same time we have from (10.37)

$$\begin{aligned}\frac{\mu}{\lambda_0}E(t) \le G(t) &\le \left[\lambda + \mu(C_0 + \mu(n+1) + K_0)\right]E(t) \\ &= \mu_0 E(t)\end{aligned} \tag{10.44}$$

where the positive constant

$$\mu_0 = a\mu^2 + b\mu + c \tag{10.45}$$

is a quadratic function of μ, and $a = n+1$, $b = C_0 + K_0$, $c = \lambda$ are independent of it. Use of (10.44) in (10.43), yields the differential inequality

$$G'(t) + \beta G(t) \le 0 \qquad t \ge 0, \tag{10.46}$$

where

$$\beta = \mu/\mu_0. \tag{10.47}$$

Multiplying (10.46) by $e^{\beta t}$ and integrating from zero to t, we get

$$G(t) \le e^{-\beta t}G(0) \qquad t \ge 0.$$

Eventually it follows from (10.44) that

$$E(t) \le Me^{-\beta t}E(0) \qquad \text{for } t \ge 0 \tag{10.48}$$

where,

$$M = \frac{\mu_0 \lambda_0}{\mu} \ge 1 \tag{10.49}$$

in virtue of equation (10.44).

10.5 Concluding Remarks

The expression for β in (10.47) with (10.45) as a function of the viscoelastic damping parameter μ shows that the decay rate is maximum for $\mu = \sqrt{\lambda/(n+1)}$. Hence from (10.42), the maximum decay rate is attained for

$$\mu = \min\left\{1/\lambda_2,\ 1/2KK_0,\ \lambda/C_0,\ \sqrt{\lambda/(n+1)}\right\}. \tag{10.50}$$

Properties of the decay rate is restricted by the lack of explicit knowledge in general, of the constants λ, λ_2, K, K_0, C_0 appearing in the expression.

This study deals with the exponential decay of the solution of the internally damped wave equation (10.4) together with boundary conditions (10.5) and (10.6) and initial conditions (10.7) in the sense of decay of the total energy according to the stated Theorem. The problem considered here is a generalization of abstract dynamical system with internally damping term. This is the main interest of this analysis, since internal structural damping is always present in actual systems (cf. Christensen [16]). The boundary conditions are standard without boundary damping. For establishing the stability theory, recourse is taken to the available methods of functional analysis, basing the main result on the necessary Sobolev spaces for the initial values of the system. Finally we conclude that systems of such type ultimately go to rest due to their own material damping property.

CHAPTER 11

BOUNDARY STABILITY OF INTERNALLY DAMPED WAVE EQUATION IN A BOUNDED DOMAIN IN $\mathbf{R}^n$ †

11.1 Introduction

The formulation of the mathematical problem in the last Chapter, has put forward one step towards the generalization of linerized wave equation to Kelvin-Voigt viscoelastic materials, where the stress is not simply proportional to strain. There are many flexible structures, the dynamics of which are complicated and non-linear in practice. Linearized models are thus sought for simplicity and more accurately to describe the physical phenomena exactly to some extent. In this Chapter, we move further to generalize linearly the mathematical formulation of the last Chapter to more realistic model for the dynamics of flexible mechanical structures. In fact, we are concerned about the uniform boundary stabilization of the mathematical problem satisfying the differential equation $y'' + \lambda y''' = c^2(\Delta y + \mu \Delta y')$, $0 < \lambda < \mu$, in a bounded domain Ω in $\mathbf{R}^n$ with smooth boundary Γ. Such equations arise in the vibrations of flexible structures possessing internal material damping and modeled by the *Standard Linear Model* of viscoelasticity, in which a linear spring is connected in series with a combination of another linear spring and a dashpot in parallel (cf. Fung [25], Rabotnov [71]) and the stress σ and strain e are related by $\sigma + \lambda\sigma' = E(e + \mu e')$. Explicit form of exponential energy decay rate is subject to investigation for the solution of the above problem with a velocity feedback at the boundary.

11.2 Mathematical Formulation

Let Ω be a bounded open connected set in $\mathbf{R}^n$ having a boundary Γ consisting of two parts Γ_0 and $\Gamma_1 \neq \emptyset$, Γ_1 being relatively open in Γ. Let x_0 be an arbitrary

† The contents of this chapter have been published in the paper ***Stability of the Boundary Stabilised Internally Damped Wave Equation*** $y'' + \lambda y''' = c^2(\Delta y + \mu \Delta y')$ ***in a Bounded Domain in*** $\mathbf{R}^n$ —Bose and Gorain, 'Indian Journal of Mathematics' Vol. 40, 1–15, (1998).

but fixed point in $\mathbf{R}^n$ and let

$$m(x) = x - x_0 \quad (x \in \mathbf{R}^n) \tag{11.1}$$

with

$$R_o = \sup_{x \in \Omega}\{|m(x)|\}, \tag{11.2}$$

such that

$$m(x).\nu(x) \leq 0 \quad \text{on} \quad \Gamma_0, \tag{11.3}$$

$$m(x).\nu(x) \geq \gamma > 0 \quad \text{on} \quad \Gamma_1 \tag{11.4}$$

where ν denotes the outward unit normal to Γ and $.$ the scalar product in $\mathbf{R}^n$. With (11.3) and (11.4) in view we further stipulate that,

(i) if $\Gamma_0 \neq \emptyset$, Γ is of class C^2 with $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$,

(ii) if $\Gamma_0 = \emptyset$, Γ is convex with $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$,
and $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$.

Let $b(x)$ be an $L^\infty(\Gamma_1)$ function satisfying $b(x) \geq b_0 > 0$ on Γ_1. We now consider the problem

$$y'' + \lambda y''' = c^2(\Delta y + \mu \Delta y') \quad \text{in} \quad \Omega \times (0, \infty), \tag{11.5}$$

subject to the mixed boundary conditions

$$y = 0 \quad \text{on} \quad \Gamma_0 \times (0, \infty), \tag{11.6}$$

$$\frac{\partial y}{\partial \nu} = -by' \quad \text{on} \quad \Gamma_1 \times (0, \infty), \tag{11.7}$$

and the initial conditions

$$y(x,0) = y_0, \quad y'(x,0) = y_1 \quad \text{and} \quad y''(x,0) = y_2, \tag{11.8}$$

where primes denote time derivatives and Δ the Laplacian in $\mathbf{R}^n$ taken in the space variables, λ, μ being small positive constants satisfying $\lambda < \mu$ and $c > 0$ a constant. In (11.7), b is of the nature of small viscous boundary damping coefficient, with b_0 as small positive constant.

Physical motivation for studying the problem (11.5) arises from the problem of vibrations of an elastic structure with internal material damping, however small always present in real materials (cf. Christensen [16]). A simple but realistic model for the latter is the so called 'standard linear material model'; in which a linear spring is connected in series with a combination of another linear spring and a dashpot in parallel (cf. Fung [25], Rabotnov [71]). In this case the differential relation connecting stress σ and strain e is given by

$$\sigma + \lambda\sigma' = E(e + \mu e'). \tag{11.9}$$

Equation (11.9) models real flexible mechanical material better than ordinary Hooke's Law. Torsional and longitudinal vibrations of a linear uniform structure leads to (11.5) in one-space dimension and equation (11.5) is its generalisation in $\mathbf{R}^n$. The case $\lambda = 0$ for the so called Kelvin-Voigt model of viscoelasticity has been treated in the previous Chapter.

We notice in the boundary conditions (11.6) and (11.7) that there exists a Neumann action on the boundary Γ_1 and zero Dirichlet action on the boundary Γ_0. In fact, the boundary Γ comprises partly of a nontrapping reflecting surface and partly an energy absorbing surface. Eventually the action on the boundary Γ_1 entails the viscous damping (velocity feedback damping) on the boundary Γ_1.

For the sake of simplicity, we substitute

$$u = y + \lambda y' \tag{11.10}$$

for the system (11.5)–(11.7). The governing differential equation (11.5) then becomes

$$u'' = c^2\Delta u + c^2(\mu - \lambda)\Delta y' \quad \text{in } \Omega \times (0, \infty) \tag{11.11}$$

and the boundary conditions (11.6)–(11.7) change to

$$u = 0 \quad \text{on} \quad \Gamma_0 \times (0,\infty), \tag{11.12}$$

$$\frac{\partial u}{\partial \nu} = -bu' \quad \text{on} \quad \Gamma_1 \times (0,\infty). \tag{11.13}$$

Evidently, the system (11.10)–(11.13) is equivalent to the system (11.5)–(11.7).

11.3 Energy of the System

To define the energy $E(t)$ of the system (11.10)–(11.13), we first multiply (11.11) by u' and integrate over Ω, which on application of Green's formula yields

$$\int_\Omega u''u'dx = c^2\int_\Gamma \left[\frac{\partial u}{\partial \nu} + (\mu - \lambda)\frac{\partial y'}{\partial \nu}\right]u'd\Gamma - c^2\int_\Omega \left[\nabla u + (\mu - \lambda)\nabla y'\right].\nabla u'dx. \tag{11.14}$$

Applying the relation (11.10) and the boundary conditions (11.7), (11.12) and (11.13), we obtain from (11.14)

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega}\left(u'^2+c^2|\nabla u|^2\right)dx &= -c^2\int_{\Gamma_1} bu'^2 d\Gamma - c^2(\mu-\lambda)\int_{\Gamma_1} by''(y'+\lambda y'')d\Gamma \\ &\quad -c^2(\mu-\lambda)\int_{\Omega}\nabla y'.\nabla(y'+\lambda\nabla y'')dx,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega}\left[u'^2+c^2|\nabla u|^2+c^2(\mu-\lambda)\lambda|\nabla y'|^2\right]dx+\frac{1}{2}c^2(\mu-\lambda)\frac{\partial}{\partial t}\int_{\Gamma_1} by'^2 d\Gamma \\ &= -c^2\int_{\Gamma_1} bu'^2 d\Gamma - c^2(\mu-\lambda)\lambda\int_{\Gamma_1} by''^2 d\Gamma - c^2(\mu-\lambda)\int_{\Omega}|\nabla y'|^2 dx. \qquad (11.15)\end{aligned}$$

At this stage, if we define the energy of the system (11.11)–(11.13) by the functional

$$E(t)=\frac{1}{2}\int_{\Omega}[u'^2+c^2|\nabla u|^2+c^2\lambda(\mu-\lambda)|\nabla y'|^2]dx+\frac{1}{2}c^2(\mu-\lambda)\int_{\Gamma_1} by'^2 d\Gamma \qquad (11.16)$$

where u and y are related by (11.10), then clearly it follows from (11.15)

$$E'(t)=-c^2\int_{\Gamma_1} bu'^2 d\Gamma - c^2(\mu-\lambda)\lambda\int_{\Gamma_1} by''^2 d\Gamma - c^2(\mu-\lambda)\int_{\Omega}|\nabla y'|^2 dx. \qquad (11.17)$$

From (11.17) it follows that $E'(t) \leq 0$, that means, the energy $E(t)$ is nonincreasing with time and hence

$$E(t) \leq E(0) \quad \text{for } t \geq 0. \qquad (11.18)$$

The third integral in (11.17) shows that energy of the system is dissipating throughout the domain due to the presence of the internal damping of the system and similar is the case from the first two integrals due to viscous boundary damping of the system.

There are several papers on the problem of boundary stability for the solution of wave equation in a bounded domain (cf. Chen [6,9], Lagnese [42,43], Lasiecka and Triggiani [46], Komornik [37], Komornik and Zuazua [38] and Lions [52], to name but few). Chen [6] first established explicitly the exponential energy decay rate for the solution of wave equation by considering certain geometries of the domain. In order to obtain stability of the wave, distributed viscous boundary damping is taken into consideration. Later, Chen [9] treated the wave equation having both distributed viscous damping and boundary damping. The theory of boundary stabilization of wave equation has been improved by Lagnese [43]. Komornik [37] obtained faster energy decay rate for such problem by constructing a special type of feedback.

11.4. Uniform Boundary Stability Result

The main result viz., the uniform exponential stability for the solution of the above system (11.10)–(11.13), is follows from ensuing Theorem.

Theorem 11.1. If for every initial values $\{y_0, y_1, y_2\}$ for which $E(0) < \infty$, then the energy $E(t)$ of the system (11.5)–(11.8) converges to zero uniformly exponentially as $t \to +\infty$. In other words, $E(t)$ satisfies the result

$$E(t) \leq Me^{-\beta t}E(0) \qquad t \geq 0, \tag{11.19}$$

for some reals $M \geq 1$ and $\beta > 0$.

The result (11.19) will be obtained after some preliminary steps.

Lemma 11.1. For every $u \in H^1(\Omega)$

$$\int_\Omega [2u.(m.\nabla)u + n|u|^2]dx = \int_\Gamma m.\nu|u|^2 d\Gamma$$

where $H^1(\Omega)$ is the classical Sobolev space of real valued function of order one.

Proof. We have

$$\begin{aligned}\int_\Omega \left[2u.(m.\nabla)u + n|u|^2\right]dx &= \int_\Omega \left[(m.\nabla|u|^2) + n|u|^2\right]dx \\ &= \int_\Omega \mathrm{div}(m|u|^2)dx \\ &= \int_\Gamma m.\nu|u|^2 d\Gamma.\end{aligned}$$

Hence the Lemma.

Lemma 11.2. If $y(x,t)$ is a regular solution of (11.5)–(11.8), then the function $\rho(t)$ defined by

$$\rho(t) = \frac{1}{2}\int_\Omega \left(|\nabla u'|^2 + c^2|\Delta u|^2\right)dx + c^2\frac{(\mu-\lambda)\lambda}{2}\int_\Omega |\Delta y'|^2 dx, \tag{11.20}$$

is nonincreasing with time.

Proof. Differentiating (11.20) with respect to t and then replacing $c^2\Delta u$ by $u'' - c^2(\mu - \lambda)\Delta y'$ from the relation (11.11), we obtain

$$\rho'(t) = \int_\Omega \left(u''\Delta u' + \nabla u'.\nabla u''\right)dx - c^2(\mu-\lambda)\int_\Omega \Delta y'\left(\Delta u' - \lambda\Delta y''\right)dx.$$

Applying Green's formula, and using the relation (11.10) and the boundary conditions (11.12)–(11.13), we get

$$\rho'(t) = -\int_{\Gamma_1} bu''^2 d\Gamma - c^2(\mu-\lambda)\int_\Omega |\Delta y'|^2 dx. \tag{11.21}$$

Thus the function $\rho(t)$ is nonincreasing for $t \geq 0$ and we have $\rho(t) \leq \rho(0)$, for $t \geq 0$.

Lemma 11.3. If $y(x,t)$ is a regular solution of (11.5)–(11.8) and ε is a small but fixed positive real, then

$$\begin{aligned}\rho_0'(t)+\rho_1'(t) \; + \; \rho_2'(t) &\leq -2E(t)+c^2(\mu-\lambda)\Big[K(\mu-\lambda)\int_\Omega |\Delta y'|^2dx + K_0\int_\Omega |\nabla y'|^2dx\Big] \\ &+ \frac{\varepsilon c^2(n-1)^2(\mu+\lambda)K_1}{2\lambda}\int_\Omega |\nabla u|^2dx - 2c^2(\mu-\lambda)\int_\Omega \Big(\nabla y'.(m.\nabla)\nabla u\Big)dx \\ &+ c^2\int_{\Gamma_1}\Big(\frac{R_0}{c^2}+\frac{b}{2\varepsilon}\Big)u'^2d\Gamma + \frac{c^2(\mu-\lambda)\lambda}{\varepsilon}\int_{\Gamma_1} by''^2d\Gamma \qquad (11.22)\end{aligned}$$

where

$$\rho_0(t) = \int_\Omega 2u'(m.\nabla u)dx, \qquad (11.23)$$

$$\rho_1(t) = \int_\Omega (n-1)uu'dx, \qquad (11.24)$$

$$\rho_2(t) = \frac{1}{2}(n+1)c^2(\mu-\lambda)\int_\Omega |\nabla y|^2dx, \qquad (11.25)$$

and the constants K, K_0, K_1 are independent of t, defined by the inequalities

$$\int_{\Gamma_0} |m.\nu||\nabla y'|^2d\Gamma \leq K\int_\Omega |\Delta y'|^2dx, \qquad (11.26)$$

$$\int_{\Gamma_1} by'^2d\Gamma \leq K_0\int_\Omega |\nabla y'|^2dx, \qquad (11.27)$$

for all $y' \in H^2_{\Gamma_0}(\Omega) = \{f \; : \; f \in H^2(\Omega), \; f=0 \text{ on } \Gamma_0\}$ and

$$\int_{\Gamma_1} bu^2d\Gamma \leq K_1\int_\Omega |\nabla u|^2dx, \qquad (11.28)$$

for all $u \in H^1_{\Gamma_0}(\Omega) = \{f \; : \; f \in H^1(\Omega), \; f=0 \text{ on } \Gamma_0\}$.

Proof. Differentiating (11.23) and (11.24) with respect to t and using (11.11), we have

$$\begin{aligned}\rho_0'(t)+\rho_1'(t) &= 2c^2\int_\Omega \big[\Delta u+(\mu-\lambda)\Delta y'\big](m.\nabla u)dx + 2\int_\Omega u'(m.\nabla u')dx \\ &+(n-1)c^2\int_\Omega \big[\Delta u+(\mu-\lambda)\Delta y'\big]udx + (n-1)\int_\Omega u'^2dx. \qquad (11.29)\end{aligned}$$

Applying Green's formula (11.29) gives

$$\begin{aligned}\rho_0'(t) \; + \; \rho_1'(t) &= 2c^2\int_\Gamma \Big[\frac{\partial u}{\partial \nu}+(\mu-\lambda)\frac{\partial y'}{\partial \nu}\Big](m.\nabla u)d\Gamma + (n-1)c^2\int_\Gamma \Big[\frac{\partial u}{\partial \nu}+(\mu-\lambda)\frac{\partial y'}{\partial \nu}\Big]ud\Gamma \\ &- \; 2c^2\int_\Omega \big[\nabla u+(\mu-\lambda)\nabla y'\big].\nabla(m.\nabla u)dx - (n-1)c^2\int_\Omega \big[\nabla u+(\mu-\lambda)\nabla y'\big].\nabla udx \\ &+ \; 2\int_\Omega u'(m.\nabla u')dx + (n-1)\int_\Omega u'^2dx.\end{aligned}$$

Since, $\nabla(m.\nabla u) = \nabla u + (m.\nabla)\nabla u$, from above after a simplication leads to

$$\begin{aligned}\rho_0'(t)+\rho_1'(t) = {} & c^2\int_\Gamma \Big[2(m.\nabla u)+(n-1)u\Big]\Big[\frac{\partial u}{\partial\nu}+(\mu-\lambda)\frac{\partial y'}{\partial\nu}\Big]d\Gamma \\ & - c^2\int_\Omega \Big[2\nabla u.(m.\nabla)\nabla u+(n+1)|\nabla u|^2\Big]dx+\int_\Omega\Big[2u'(m.\nabla)u'+(n-1)u'^2\Big]dx. \\ & - c^2(\mu-\lambda)\int_\Omega\Big[2\nabla y'.(m.\nabla)\nabla u+(n+1)\nabla u.\nabla y'\Big]dx. \qquad (11.30)\end{aligned}$$

Applying Lemma 11.1, (11.30) yields

$$\begin{aligned}\rho_0'(t)+\rho_1'(t) = {} & c^2\int_\Gamma \Big[2(m.\nabla u)+(n-1)u\Big]\Big[\frac{\partial u}{\partial\nu}+(\mu-\lambda)\frac{\partial y'}{\partial\nu}\Big]d\Gamma+\int_\Gamma m.\nu\Big(u'^2-c^2|\nabla u|^2\Big)d\Gamma \\ & -\int_\Omega\Big(u'^2+c^2|\nabla u|^2\Big)dx-2c^2(\mu-\lambda)\int_\Omega\Big(\nabla y'.(m.\nabla)\nabla u\Big)dx \\ & -\frac{1}{2}c^2(\mu-\lambda)(n+1)\frac{\partial}{\partial t}\int_\Omega|\nabla y|^2dx-c^2(\mu-\lambda)(n+1)\lambda\int_\Omega|\nabla y'|^2dx.\end{aligned}$$

With the help of the energy equation (11.16) and relation (11.25), the above becomes

$$\begin{aligned}\rho_0'(t)+\rho_1'(t)+\rho_2'(t) \le {} & c^2\int_\Gamma \Big[2(m.\nabla u)+(n-1)u\Big]\Big[\frac{\partial u}{\partial\nu}+(\mu-\lambda)\frac{\partial y'}{\partial\nu}\Big]d\Gamma \\ & +\int_\Gamma m.\nu\Big(u'^2-c^2|\nabla u|^2\Big)d\Gamma+c^2(\mu-\lambda)\int_{\Gamma_1}by'^2d\Gamma \\ & -2c^2(\mu-\lambda)\int_\Omega\Big[\nabla y'.(m.\nabla)\nabla u\Big]dx-2E(t). \qquad (11.31)\end{aligned}$$

Applying the inequality (11.27), the relation (11.31) gives

$$\begin{aligned}\rho_0'(t)+\rho_1'(t)+\rho_2'(t) \le {} & c^2\int_\Gamma \Big[2(m.\nabla u)+(n-1)u\Big]\Big[\frac{\partial u}{\partial\nu}+(\mu-\lambda)\frac{\partial y'}{\partial\nu}\Big]d\Gamma \\ & +\int_\Gamma m.\nu\Big(u'^2-c^2|\nabla u|^2\Big)d\Gamma+c^2K_0(\mu-\lambda)\int_\Omega|\nabla y'|^2dx \\ & -2c^2(\mu-\lambda)\int_\Omega\Big[\nabla y'.(m.\nabla)\nabla u\Big]dx-2E(t). \qquad (11.32)\end{aligned}$$

Now we consider the boundary integrals in (11.32) and set

$$X = c^2\int_\Gamma \Big[2(m.\nabla u)+(n-1)u\Big]\Big[\frac{\partial u}{\partial\nu}+(\mu-\lambda)\frac{\partial y'}{\partial\nu}\Big]d\Gamma \qquad (11.33)$$

and

$$Y = \int_\Gamma m.\nu\Big(u'^2-c^2|\nabla u|^2\Big)d\Gamma. \qquad (11.34)$$

In the sequel, we use the following inequalities

$$|ab| \le \frac{1}{2}\Big(\alpha a^2+\frac{1}{\alpha}b^2\Big), \text{ for any positive real } \alpha, \qquad (11.35)$$

$$(a+b)^2 \le 2(a^2+b^2), \qquad (11.36)$$

and combination of (11.35) and (11.36) gives

$$|(a+b)c| \leq \alpha(a^2+b^2) + \frac{c^2}{2\alpha}. \tag{11.37}$$

Since $y=0$ on Γ_0, $\nabla y = \nu(\partial y/\partial\nu)$ and $|\nabla y|^2 = |(\partial y/\partial \nu)|^2$ on Γ_0 and also since $u=0$ on Γ_0, $\nabla u = \nu(\partial u/\partial \nu)$ and $|\nabla u|^2 = |(\partial u/\partial\nu)|^2$ on Γ_0. Applying these (11.33) can be written as

$$\begin{aligned}
X &= c^2\int_{\Gamma_0} 2(m.\nu)\Big[\Big(\frac{\partial u}{\partial \nu}\Big)^2 + (\mu-\lambda)\frac{\partial u}{\partial \nu}\frac{\partial y'}{\partial \nu}\Big]d\Gamma \\
&\quad + c^2\int_{\Gamma_1}\big[2(m.\nabla u)+(n-1)u\big]\Big[\frac{\partial u}{\partial\nu}+(\mu-\lambda)\frac{\partial y'}{\partial \nu}\Big]d\Gamma \\
&\leq c^2\int_{\Gamma_0}2(m.\nu)\Big(\frac{\partial u}{\partial\nu}\Big)^2 d\Gamma + c^2(\mu-\lambda)\int_{\Gamma_0}|m.\nu|\Big[\frac{1}{\mu-\lambda}\Big(\frac{\partial u}{\partial \nu}\Big)^2+(\mu-\lambda)\Big(\frac{\partial y'}{\partial\nu}\Big)^2\Big]d\Gamma \\
&\quad + c^2\int_{\Gamma_1} b\Big[\varepsilon\big(4|m.\nabla u|^2+(n-1)^2u^2\big)+\frac{u'^2}{2\varepsilon}\Big]d\Gamma \\
&\quad + c^2(\mu-\lambda)\int_{\Gamma_1} b\Big[\frac{\varepsilon}{2\lambda}\big(4|m.\nabla u|^2+(n-1)^2u^2\big)+\frac{\lambda y''^2}{\varepsilon}\Big]d\Gamma
\end{aligned} \tag{11.38}$$

where we have used the inequalities (11.35) and (11.37) and the boundary conditions (11.7) and (11.13). Since $m.\nu \leq 0$ on Γ_0, we have $m.\nu + |m.\nu| = 0$ on Γ_0 and hence from (11.38),

$$\begin{aligned}
X &\leq c^2\int_{\Gamma_0}(m.\nu)\Big(\frac{\partial u}{\partial\nu}\Big)^2 d\Gamma + c^2(\mu-\lambda)^2\int_{\Gamma_0}|m.\nu||\nabla y'|^2 d\Gamma + 2\varepsilon c^2 R_0^2\frac{\mu+\lambda}{\lambda}\int_{\Gamma_1} b|\nabla u|^2 d\Gamma \\
&\quad + \varepsilon c^2(n-1)^2\frac{\mu+\lambda}{2\lambda}\int_{\Gamma_1} bu^2 d\Gamma + \frac{c^2}{2\varepsilon}\int_{\Gamma_1} bu'^2 d\Gamma + c^2\frac{(\mu-\lambda)\lambda}{\varepsilon}\int_{\Gamma_1} by''^2 d\Gamma.
\end{aligned} \tag{11.39}$$

Again from (11.34), we can have similarly

$$Y \leq -c^2\int_{\Gamma_0}(m.\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\Gamma + \int_{\Gamma_1}(m.\nu)u'^2 d\Gamma - c^2\int_{\Gamma_1}(m.\nu)|\nabla u|^2 d\Gamma. \tag{11.40}$$

By the inequalities (11.26) and (11.28), we have from (11.39) and (11.40)

$$\begin{aligned}
X + Y &\leq c^2(\mu-\lambda)^2 K\int_\Omega |\Delta y'|^2 dx + c^2\int_{\Gamma_1}\Big[2\varepsilon b R_0^2\frac{\mu+\lambda}{\lambda}-\gamma\Big]|\nabla u|^2 d\Gamma \\
&\quad + \varepsilon c^2(n-1)^2\frac{\mu+\lambda}{2\lambda}K_1\int_\Omega|\nabla u|^2 dx + c^2\int_{\Gamma_1}\Big(\frac{R_0}{c^2}+\frac{b}{2\varepsilon}\Big)u'^2 d\Gamma + c^2\frac{(\mu-\lambda)\lambda}{\varepsilon}\int_{\Gamma_1} by''^2 d\Gamma.
\end{aligned}$$

Taking $\varepsilon \leq (\lambda\gamma/2bR_0^2)(\mu+\lambda)^{-1}$, the above ultimately yields

$$\begin{aligned}
X+Y &\leq c^2(\mu-\lambda)^2K\int_\Omega|\Delta y'|^2dx + \varepsilon c^2(n-1)^2\frac{\mu+\lambda}{2\lambda}K_1\int_{\Gamma_1}|\nabla u|^2 dx \\
&\quad + c^2\int_{\Gamma_1}\Big(\frac{R_0}{c^2}+\frac{b}{2\varepsilon}\Big)u'^2 d\Gamma + c^2\frac{(\mu-\lambda)\lambda}{\varepsilon}\int_{\Gamma_1} by''^2 d\Gamma.
\end{aligned} \tag{11.41}$$

Using the relation (11.41), the Lemma follows immediately from (11.32).

We are now ready to establish the main result (11.19) for the solution of the system (11.10)–(11.13).

Proof of the Theorem. We now introduce a function $G(t)$ for all $t \geq 0$ by

$$G(t) = E(t) + \varepsilon\Big[\rho_0(t) + \rho_1(t) + \rho_2(t)\Big] + \varepsilon K(\mu - \lambda)\rho(t) \tag{11.42}$$

and define a constant $K_2 > 1$ independent of t, such that

$$\int_\Omega u^2 dx \leq K_2 \int_\Omega |\nabla u|^2 dx \tag{11.43}$$

for all $u \in H^1_{\Gamma_0}(\Omega)$.

Inequality (11.43) is due to Poincare and inequalities (11.26), (11.27), (11.28) follow from combination of Poincare inequality and Trace inequality (cf. Aubin [1]).

Now we have from (11.23),

$$|\rho_0(t)| \leq \frac{R_0}{c}\int_\Omega \Big(u'^2 + c^2|\nabla u|^2\Big)dx \leq \frac{2R_0}{c}E(t) \tag{11.44}$$

and

$$|\rho_1(t)| \leq \frac{n-1}{2c}\int_\Omega (u'^2 + c^2u^2)dx \leq \frac{n-1}{c}K_2E(t) \tag{11.45}$$

by the use of (11.43). Also there exists a constant $K_3 > 0$, such that we can write

$$0 \leq \rho_2(t) \leq K_3E(t). \tag{11.46}$$

With the help of (11.44)–(11.46), it follows from (11.42) that

$$\begin{aligned}\Big[1 - \varepsilon\Big(\frac{2R_0}{c} + \frac{n-1}{c}K_2\Big)\Big]E(t) \;\leq\; G(t) \;\leq\; &\Big[1 + \varepsilon\Big(\frac{2R_0}{c} + \frac{n-1}{c}K_2 + K_3\Big)\Big]E(t)\\ &+\; \varepsilon K(\mu - \lambda)\rho(t).\end{aligned} \tag{11.47}$$

Since $\varepsilon > 0$ is small, taking $\varepsilon \leq \Big(2R_0/c + K_2(n-1)/c + K_3\Big)^{-1}$ we have from (11.47),

$$\varepsilon K_3E(t) \leq G(t) \leq K_4E(t) + \varepsilon K(\mu - \lambda)\rho(t) \tag{11.48}$$

where

$$K_4 = 1 + \varepsilon\Big[\frac{2R_0}{c} + \frac{n-1}{c}K_2 + K_3\Big].$$

Now $0 < E(t) \leq E(0)$ and $0 < \rho(t) \leq \rho(0)$ in virtue of (11.18) and Lemma 11.2. Therefore there exists a constant $K_5 > 0$ such that

$$\rho(t) \leq K_5E(t). \tag{11.49}$$

Thus we have from (11.48),

$$\varepsilon K_3 E(t) \leq G(t) \leq K_6 E(t) \tag{11.50}$$

where,

$$K_6 = K_4 + \varepsilon(\mu - \lambda)KK_5. \tag{11.51}$$

We can assert that $K_5 < \infty$, provided that $\sup\{\rho(t)/E(t)\}$ is finite for $t \geq 0$. Invoking Poincare inequality

$$\int_\Omega |\nabla y'|^2 dx \leq K^* \int_\Omega |\Delta y'|^2 dx \tag{11.52}$$

for all $y' \in H^2_{\Gamma_0}(\Omega)$, where K^* is a positive number (which can be taken as large as we please), we have from (11.17) and (11.21),

$$E'(t) \geq K^* \rho'(t) + \left[K^* \int_{\Gamma_1} bu''^2 d\Gamma - c^2 \int_{\Gamma_1} bu'^2 d\Gamma - c^2(\mu - \lambda)\lambda \int_{\Gamma_1} by''^2 d\Gamma\right].$$

Here, the integrals within the brackets arise from small viscous boundary damping along Γ_1 only, and can be suppressed by taking K^* large enough to write

$$E'(t) \geq K^{**} \rho'(t) \tag{11.53}$$

where $K^{**} \geq K^*$. Since $E'(t)$ and $\rho'(t)$ are both negative, we conclude that $\rho(t)$ decreases at a much faster rate than $E(t)$ and the assertion follows.

Now we proceed with the differentiation of (11.42) with respect to t, to obtain

$$G'(t) = E'(t) + \varepsilon[\rho_0'(t) + \rho_1'(t) + \rho_2'(t)] + \varepsilon K(\mu - \lambda)\rho'(t). \tag{11.54}$$

Using the relations (11.17), (11.21) and Lemma 11.3, we obtain from (11.54)

$$\begin{aligned} G'(t) \ \leq \ & -2\varepsilon E(t) + c^2(\mu - \lambda)(\varepsilon K_0 - 1) \int_\Omega |\nabla y'^2|^2 dx + \varepsilon^2 c^2 \frac{(n-1)^2(\mu + \lambda)K_1}{2\lambda} \int_\Omega |\nabla u|^2 dx \\ & -2c^2\varepsilon(\mu - \lambda) \int_\Omega [\nabla y'.(m.\nabla)\nabla u] dx. + c^2 \int_{\Gamma_1} \left(\frac{\varepsilon R_0}{c^2} - \frac{b}{2}\right) u'^2 d\Gamma. \end{aligned} \tag{11.55}$$

We note that,

$$\begin{aligned} \left|2 \int_\Omega [\nabla y'.(m.\nabla)\nabla u] dx\right| \ & \leq \ \int_\Omega \left[\frac{1}{2\varepsilon}|\nabla y'|^2 + 2\varepsilon|(m.\nabla)\nabla u|^2\right] dx \\ & \leq \ \frac{1}{2\varepsilon} \int_\Omega |\nabla y'|^2 dx + 2\varepsilon K_7 \int_\Omega |\Delta u|^2 dx \end{aligned} \tag{11.56}$$

by the use of (11.35), and K_7 is a positive constant independent of t, defined by

$$\int_\Omega |(m.\nabla)\nabla u|^2 dx \leq K_7 \int_\Omega |\Delta u|^2 dx. \tag{11.57}$$

The inequality (11.57) is due to the fact that the expressions $m_k(\partial^2 u/\partial x_k \partial x_j)$ $(j,k = 1,2,...,n)$ in the integrand of left hand side of (11.57) can be transformed to the form $M_k(\partial^2 u/\partial x_k^2)$ (with usual summation convention of repeated indices) by suitable orthogonal transformation of the axes x_j and Δu is invarient with respect to orthogonal transformation (cf. [20]).

Taking $\varepsilon \leq bc^2/2R_0$ also and using (11.56), we have from (11.55)

$$G'(t) \leq -2\varepsilon E(t) + c^2(\mu-\lambda)(\varepsilon K_0 - \frac{1}{2}) \int_\Omega |\nabla y'|^2 dx + \varepsilon^2 c^2 (n-1)^2 \frac{\mu+\lambda}{2\lambda} K_1 \int_\Omega |\nabla u|^2 dx$$
$$+ 2\varepsilon^2 c^2 (\mu-\lambda) K_7 \int_\Omega |\Delta u|^2 dx$$

Use of the relations (11.16) and (11.20), the above can be written as

$$G'(t) \leq \varepsilon^2 (n-1)^2 \frac{\mu+\lambda}{\lambda} K_1 E(t) + 4\varepsilon^2(\mu-\lambda)K_7 \rho(t) - 2\varepsilon E(t), \tag{11.58}$$

where we take $\varepsilon \leq 1/2K_0$. By the relation (11.49), we have from (11.58)

$$G'(t) \leq \varepsilon \Big[\varepsilon \Big(\frac{(n-1)^2(\mu+\lambda)}{\lambda} K_1 + 4(\mu-\lambda)K_5K_7\Big) - 1\Big] E(t) - \varepsilon E(t). \tag{11.59}$$

As ε is small, we can further suppose that $\varepsilon \leq \Big[(n-1)^2(\mu+\lambda)K_1/\lambda + 4(\mu-\lambda)K_5K_7\Big]^{-1}$, and it then follows from (11.59) that

$$G'(t) + \varepsilon E(t) \leq 0. \tag{11.60}$$

With the help of (11.50), the realtion (11.60) leads the differential inequality

$$G'(t) + \beta G(t) \leq 0, \tag{11.61}$$

where $\beta = \varepsilon/K_6$. The constant K_6 is given by (11.51) and the fixed positive number ε is the smallest of $(\gamma\lambda/2bR_0^2)(\mu+\lambda)^{-1}$, $\big(2R_0/c + (n-1)K_2/c + K_3\big)^{-1}$, $bc^2/2R_0$, $1/2K_0$ and $\big((n-1)^2(\mu+\lambda)K_1/\lambda + 4(\mu-\lambda)K_5K_7\big)^{-1}$. Integrating (11.61) from zero to t, we obtain the result

$$G(t) \leq e^{-\beta t} G(0) \qquad \text{for } t \geq 0. \tag{11.62}$$

Again by the use of (11.50), it follows finally from (11.62) that

$$E(t) \leq M e^{-\beta t} E(0) \qquad \text{for } t \geq 0, \tag{11.63}$$

where $M = K_6/\varepsilon K_3$. This completes the proof.

11.5 Concluding Remarks

The result (11.63) gives explicitly the exponential energy decay rate for the solution of the system (11.5)–(11.8). Since the formulation of the problem (11.5) is more general than that of wave equation, the result (11.63) can be realised for a flexible elastic system with internal damping satisfying the model equation (11.5), such as in the vibrations of beams and plates. For example (cf. Rayleigh [76]), in the longitudinal vibrations of a beam of length ℓ, the strain e in equation (11.9) equals $\partial y/\partial x$, where y is the displacement along the beam, and the equation of motion is

$$\rho y'' = \frac{\partial \sigma}{\partial x}. \tag{11.64}$$

With the strain-stress relation (11.9), (11.64) leads to equation (11.5) in one dimension, where $c^2 = E/\rho$, where ρ is density and E the Young's modulas of the beam. If the end $x = 0$ is clamped, $y(0,t) = 0$ and if the end $x = \ell$ is free, $(\partial y/\partial x)(\ell,t) = 0$. So if a damping force proportional to y' is operating on this end, then we have $(\partial y/\partial x)(\ell,t) = -by'(\ell,t)$, $b > 0$. The same set of equations holds for torsional vibrations of the beam. Similar is the case for large rectangular panels or strips. In multi-dimensions, we may think of shear wave propogation in infinite or layered semi-infinite solids (cf. Ewing *et al.* [21]). The analysis presented here estimates decay rate of energy for the solution of such problems in a general manner.

CHAPTER 12

CONCLUSIONS

In this Thesis, we have made a presentation to study, at an advanced level of rigor, a unified treatment of current methodologies for design and analysis of controllability and stability of vibrating elastic structures. Mathematical control theory for distributed parameter system is currently under extensive development in view of application to vibration control of structural elements. Major goals of this Thesis have been the investigation and presentation of techniques for analyzing the exact controllability and uniform stabilization of torsional and flexural vibrations of flexible hybrid structures. The hybrid models which are totally free at one end and appropriate control force or torque is applicable at the other end is the focal point for the suppressing the vibrations of the overall system following prescribed initial data, with out applying any constraint at the free end. The investigation has put forward estimated least time for exact controllability in the framework of HUM, due to Lions [52]. For studying more realistic linearized models of vibration, internal damping of the material is taken into account in the Thesis. In this context, we point out that uniform stability for undamped wave equation is impossible unless there is a boundary feedback (viscous boundary damping) on some part of the boundary. In the Thesis we have examined the nature of uniform stabilization of an internally damped wave equation in a bounded domain Ω in R^n without boundary feedback, owing to the fact that, inherent small material damping of Kelvin-Voigt model of viscoelasticity (cf. Christensen [16]) is always present in real materials.

Controllability and stabilization of vibrating elastic structures is an active area of reasearch and there is a great deal more to be done. In this Thesis, we have focussed on two types of vibrations namely, torsional and flexural vibrations governed by wave equation and Euler-Bernoulli beam equation respectively. There are other theories of vibrations of elastic beams and plates such as, Timoshenko beam, Rayleigh beam and coupled torsional and flexural vibrations. Applications from the diverse point of view, techniques and associated similar analyses are needed to be explored for the vibrations of linked beams or multimember frames with controls at the joints, beams with lumped mass at end, plates of various shapes, thin cylindrical shells, loaded shells or plates and highly flexible systems. The approach need not be restricted to vibrating elastic systems but extended to other multidimensional dynamical systems. While we dealt with approximated numercal result in one case, our method can be transcribed to the other cases without much difficulty. The problem faced in the tolerance of the results could be removed by employing finite element method and leaves scope for further work.

From the mathematical point of view, the linearized model equations of different types of vibrations can be modified to more realistic linear models like the *standard linear model* of viscoelasticity. The faster uniform energy decay rates can be looked into under this concept with boundary feedback control or undamped boundary. The interesting fact that there are types of mechanical damping having non-linear characteristics in nature, the extension to non-linear boundary damping or stabilizer is a crucial consequence for groundwork of future studies in vibrating control system. Vibrating structural elements with accurate mathematical models are subject to investigation in respect of boundary control of vibrations of such elements.

It should however be mentioned that there are many types of flexible space structures, the dynamics of which are complicated and non-linear in practice. The precise form mathematical models never describe the physical phenomena exactly. Linearized models are fabricated by analysists purely for simplicity and design, and describe the process only approximately. Due to complexity of dynamical systems and need for control with high-quality performance, it is necessary to develop a better understanding of the fundamental limitations and capabalities of control system design. These types of non-linear systems require sophisticated control strategies to achieve acceptable performance within the uncertain environment in which they operate. Typically, the various parameters of the systems are not known precisely *a priori* ; however crude estimates are available in the literature. Due to lack of accurate model of dynamics and exact measurements of parameters required in the theory, the question of robustness (cf. Dahleh and Diaz-Bobillo [18]) of exact theory may be addressed to theoretically.

More research will be needed to develop systematic procedures for designing various vibrating control systems which help to achieve a target performance in the presence of uncertainty. However in more complicated systems, analytical solution is rarely possible and it is necessary to resort to numerical techniques. The idea of robustness is essential for successful applications to exact dynamical systems as they are very different from linear systems having only uncertain parameters. A major problem remains to qualify robustness for these class of system.

Although there are greater scopes for treating extensively non-linear dynamical systems, the systematic investigation on the linearized models are not less elegant. Theoretical understanding endures to put forward the ways of generalization of the linearized system and the importance of the research in linear models will never be impeded. A mathematician always demands mathematical validity for a procedure that would always mathematically work. We feel consequently that in the years to come, there will be a confluence of theories and techniques for better understanding of the variety of dynamical systems, we have described in this Thesis.

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17-05-99.

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